1. Problem 7.9

a) The received signal may be expressed as

\[ r(t) = \begin{cases} 
  n(t) & \text{if } s_0(t) \text{ was transmitted} \\
  A + n(t) & \text{if } s_1(t) \text{ was transmitted}
\end{cases} \]

Assuming that \( s(t) \) has unit energy, then the sampled outputs of the crosscorrelators are

\[ r = s_m + n, \quad m = 0, 1 \]

where \( s_0 = 0 \), \( s_1 = A\sqrt{T} \) and the noise term \( n \) is a zero-mean Gaussian random variable with variance

\[ \sigma_n^2 = E \left[ \frac{1}{\sqrt{T}} \int_0^T n(t) dt \frac{1}{\sqrt{T}} \int_0^T n(\tau) d\tau \right] \]

\[ = \frac{1}{T} \int_0^T \int_0^T E [n(t)n(\tau)] dtd\tau \]

\[ = \frac{N_0}{2T} \int_0^T \int_0^T \delta(t-\tau) dtd\tau = \frac{N_0}{2} \]

The probability density function for the sampled output is

\[ f(r|s_0) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{r^2}{N_0}} \]

\[ f(r|s_1) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r-A\sqrt{T})^2}{N_0}} \]

Since the signals are equally probable, the optimal detector decides in favor of \( s_0 \) if

\[ \text{PM}(r, s_0) = f(r|s_0) > f(r|s_1) = \text{PM}(r, s_1) \]

otherwise it decides in favor of \( s_1 \). The decision rule may be expressed as

\[ \frac{\text{PM}(r, s_0)}{\text{PM}(r, s_1)} = e^{\frac{(r-A\sqrt{T})^2}{N_0}} e^{-\frac{(2r-A\sqrt{T})A\sqrt{T}}{N_0}} \]

\[ \overset{s_0}{\underset{s_1}{\geq}} 1 \]
or equivalently

\[
r \geq \frac{s_1}{s_0}
\]

The optimum threshold is \( \frac{1}{2} A \sqrt{T} \).

b) The average probability of error is

\[
P(e) = \frac{1}{2} P(e|s_0) + \frac{1}{2} P(e|s_1)
\]

\[
= \frac{1}{2} \int_{\frac{1}{2} A \sqrt{T}}^{\infty} f(r|s_0)dr + \frac{1}{2} \int_{-\infty}^{\frac{1}{2} A \sqrt{T}} f(r|s_1)dr
\]

\[
= \frac{1}{2} \int_{\frac{1}{2} A \sqrt{T}}^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{r^2}{N_0}} dr + \frac{1}{2} \int_{-\infty}^{\frac{1}{2} A \sqrt{T}} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r-A \sqrt{T})^2}{2N_0}} dr
\]

\[
= \frac{1}{2} \int_{\frac{1}{2} A \sqrt{T}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{r^2}{2N_0}} dr + \frac{1}{2} \int_{-\infty}^{\frac{1}{2} A \sqrt{T}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(r-A \sqrt{T})^2}{2N_0}} dr
\]

\[
= Q \left[ \frac{1}{2} \sqrt{\frac{2}{N_0} A \sqrt{T}} \right] = Q \left[ \sqrt{\text{SNR}} \right]
\]

where

\[
\text{SNR} = \frac{\frac{1}{2} A^2 T}{N_0}
\]

Thus, the on-off signaling requires a factor of two more energy to achieve the same probability of error as the antipodal signaling.

2. Problem 7.10

Since the rate of transmission is \( R = 10^5 \) bits/sec, the bit interval \( T_b \) is \( 10^{-5} \) sec. The probability of error in a binary PAM system is

\[
P(e) = Q \left[ \sqrt{\frac{2E_b}{N_0}} \right]
\]

where the bit energy is \( E_b = A^2 T_b \). With \( P(e) = P_2 = 10^{-6} \), we obtain

\[
\sqrt{\frac{2E_b}{N_0}} = 4.75 \implies E_b = \frac{4.75^2 N_0}{2} = 0.112813
\]
Thus

\[ A^2 T_b = 0.112813 \implies A = \sqrt{0.112813 \times 10^5} = 106.21 \]

3. Problem 7.11

a) For a binary PAM system for which the two signals have unequal probability, the optimum detector is

\[
\begin{align*}
\left| r \right|_{s_1} &> \frac{N_0}{4\sqrt{E_b}} \ln \frac{1-p}{p} = \eta \\
\left| r \right|_{s_2} &< \frac{N_0}{4\sqrt{E_b}} \ln \frac{1-p}{p} = \eta
\end{align*}
\]

The average probability of error is

\[
P(e) = P(e|s_1)P(s_1) + P(e|s_2)P(s_2)
\]

\[
= pP(e|s_1) + (1-p)P(e|s_2)
\]

\[
= p \int_{-\infty}^{\eta_1} f(r|s_1)dr + (1-p) \int_{\eta_1}^{\infty} f(r|s_1)dr
\]

\[
= p \int_{-\infty}^{\eta_1} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r-\sqrt{2E_b})^2}{N_0}} dr + (1-p) \int_{\eta_1}^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r+\sqrt{2E_b})^2}{N_0}} dr
\]

where

\[
\eta_1 = -\sqrt{\frac{2E_b}{N_0}} + \eta \sqrt{\frac{2}{N_0}} \quad \eta_2 = \sqrt{\frac{2E_b}{N_0}} + \eta \sqrt{\frac{2}{N_0}}
\]

Thus,

\[
P(e) = pQ\left[\sqrt{\frac{2E_b}{N_0}} - \eta \sqrt{\frac{2}{N_0}}\right] + (1-p)Q\left[\sqrt{\frac{2E_b}{N_0}} + \eta \sqrt{\frac{2}{N_0}}\right]
\]

b) If \( p = 0.3 \) and \( \frac{E_b}{N_0} = 10 \), then

\[
P(e) = 0.3Q[4.3774] + 0.7Q[4.5668] = 0.3 \times 6.01 \times 10^{-6} + 0.7 \times 2.48 \times 10^{-6}
\]

\[
= 3.539 \times 10^{-6}
\]
If the symbols are equiprobable, then

\[ P(e) = Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] = Q[\sqrt{2}] = 3.88 \times 10^{-6} \]

4. Problem 7.12

a) The optimum threshold is given by

\[ \eta = \frac{N_0}{4\sqrt{\mathcal{E}_b}} \ln \frac{1 - p}{p} = \frac{N_0}{4\sqrt{\mathcal{E}_b}} \ln 2 \]

b) The average probability of error is (\( \eta = \frac{N_0}{4\sqrt{\mathcal{E}_b}} \ln 2 \))

\[
P(e) = p(a_m = -1) \int_{\eta}^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r+\sqrt{\mathcal{E}_b})^2}{N_0}} dr + p(a_m = 1) \int_{-\infty}^{\eta} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r-\sqrt{\mathcal{E}_b})^2}{N_0}} dr
\]

\[
= \frac{2}{3} Q\left[\frac{\eta + \sqrt{\mathcal{E}_b}}{\sqrt{N_0}/2}\right] + \frac{1}{3} Q\left[\frac{\sqrt{\mathcal{E}_b} - \eta}{\sqrt{N_0}/2}\right]
\]

\[
= \frac{2}{3} Q\left[\sqrt{\frac{2N_0/\mathcal{E}_b \ln 2}{4}}\right] + \sqrt{\frac{2\mathcal{E}_b}{N_0}} + \frac{1}{3} Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}} - \sqrt{\frac{2N_0/\mathcal{E}_b \ln 2}{4}}\right]
\]