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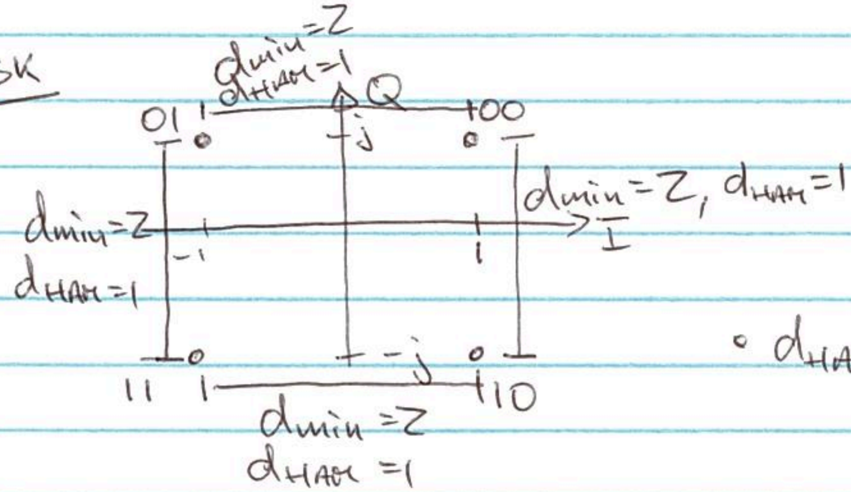
ECE 460

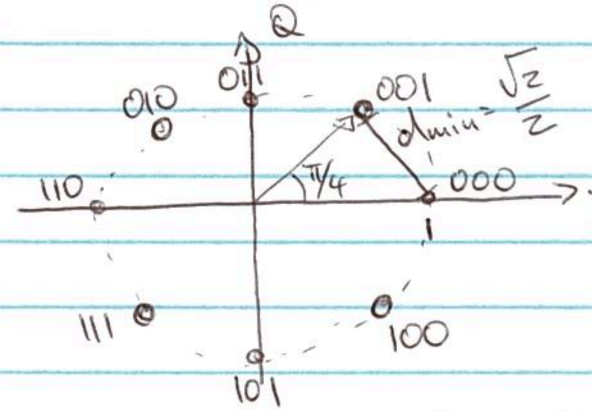
HW 9 - Solution

1. a) <u>QPSK</u>



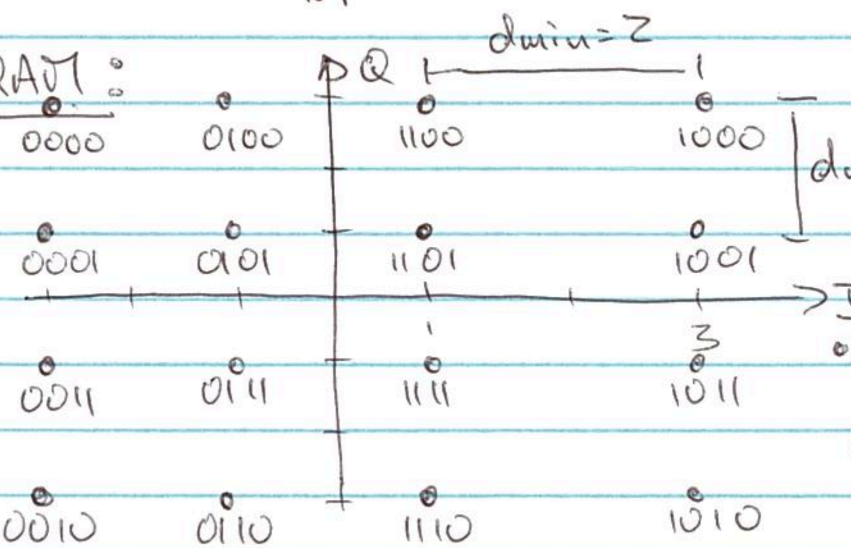
• d_{HAM} - Hamming distance: number of bit differenc

b) <u>8-PSK</u>



• adjacent symbo differ in exactly one bit ($d_{HAM}=1$)

c) <u>16 QAM:</u>



• symbols separated by $d_{min}=2$, ha $d_{HAM}=1$.

<u>Note:</u> these answers are <u>not</u> unique.

Problem 5.4

a) $Pr[b_3=0] = Pr[b_1 = b_2]$ | If $b_1=b_2 \rightarrow b_1 \oplus b_2 = 0$ Then $b_1 \oplus b_2 \oplus b_3 = 0$ only if $b_2=0$

$= Pr[b_1=0 \text{ and } b_2=0] + Pr[b_1=1 \text{ and } b_2=1]$

$= Pr[b_1=0] \cdot Pr[b_2=0] + Pr[b_1=1] \cdot Pr[b_2=1]$

$= 0.8 \cdot (1-0.9) + (1-0.8) \cdot 0.9 = \underline{\underline{0.26}}$

b) Given: $L_i = LLR(b_i)$ $i=1,2,3$

$= \ln \left(Pr[b_i=0] / Pr[b_i=1] \right)$

$= \ln \left(Pr[b_i=0] / (1 - Pr[b_i=0]) \right)$

$$\Rightarrow e^{L_i} = \frac{Pr[b_i=0]}{1 - Pr[b_i=0]}$$

1st bit: $e^{L_1} = \dfrac{Pr[b_1=0]}{1 - Pr[b_1=0]} = \dfrac{0.8}{0.2} = 4$

$\Rightarrow L_1 = \ln(4)$

2nd bit: $e^{L_2} = \dfrac{Pr[b_2=0]}{1 - Pr[b_2=0]} = \dfrac{0.1}{0.9} = \dfrac{1}{9}$

$\Rightarrow L_2 = \ln(1/9) = -\ln(9)$

3rd bit: $e^{L_3} = \dfrac{Pr[b_3=0]}{1 - Pr[b_3=0]} = \dfrac{0.26}{0.74} = \dfrac{13}{37}$

$\Rightarrow L_3 = \ln(13/37)$

Also: $e^{L_3} = \frac{Pr[b_3=0]}{1-Pr[b_3=0]} = \frac{Pr[b_1=0]\cdot Pr[b_2=0] + (1-Pr[b_1=0])(1-Pr[b_2=0])}{Pr[b_1=0]\cdot(1-Pr[b_2=0]) + Pr[b_2=0](1-Pr[b_1=0])}$

$$= \frac{\frac{Pr[b_1=0]}{1-Pr[b_1=0]} \cdot \frac{Pr[b_2=0]}{1-Pr[b_2=0]} + 1}{\frac{Pr[b_1=0]}{1-Pr[b_1=0]} + \frac{Pr[b_2=0]}{1-Pr[b_2=0]}}$$

$$= \frac{e^{L_1}\cdot e^{L_2} + 1}{e^{L_1} + e^{L_2}}$$

$$\Rightarrow L_3 = \ln\left(\frac{e^{L_1+L_2}+1}{e^{L_1}+e^{L_2}}\right)$$

5.5 a) The output Y_m for the mth channel use is Bernoulli (conditioned on $X=0$) with

$$P[Y_m = 1 \mid X=0] = a$$

Let $Z = Y_1 + Y_2 + \dots Y_n$ denote the number of 1's observed in n channel uses

$\Rightarrow$ Z is binomial (conditioned on $X=0$)

$$P(Z=z \mid X=0) = \binom{n}{z} a^z \cdot (1-a)^{n-z}$$

b) Receiver says 1 if $Z > \lfloor n/2 \rfloor$

$$\Rightarrow P_e = P(Z > \lfloor n/2 \rfloor \mid X=0) = \sum_{z=\lfloor n/2 \rfloor + 1}^{n} \binom{n}{z} a^z \cdot (1-a)^{n-z}$$

with $a=0.1$ and $n=5$: $P_e = 0.0086$

$\rightarrow$

c) $$\Pr[X=0 \mid Z=m] = \frac{\Pr[Z=m \mid X=0]\cdot P[X=0]}{P[Z=m]}$$

$$= \frac{\Pr[Z=m \mid X=0]\cdot P[X=0]}{\Pr[Z=m \mid X=0]\cdot P[X=0] + \Pr[Z=m \mid X=1]\Pr[X=1]}$$

with: $P[X=0] = P[X=1] = \frac{1}{2}$

$$P[Z=m \mid X=0] = \binom{n}{m} a^m \cdot (1-a)^{n-m}$$

$$P[Z=m \mid X=1] = \binom{n}{m} (1-a)^m \cdot a^{n-m}$$

$$\Pr[X=0 \mid Z=m] = \frac{a^m \cdot (1-a)^{n-m}}{a^m (1-a)^{n-m} + a^{n-m}(1-a)^m}$$

$$= \frac{1}{1 + \left(\frac{1-a}{a}\right)^{2m-n}}$$

m	$P(X=0 \mid Z=m)$	
0	1.000	
1	0.9986	$X=0$ most likely
2	0.9000	
3	0.1000	
4	0.0014	$X=1$ most likely
5	0.0000	

d) with $\Pr[X=0] = 1 - \Pr[X=1] = 0.9$

$$\Pr[X=0 \mid Z=m] = \frac{0.9\, a^m (1-a)^{n-m}}{0.9\, a^m (1-a)^{n-m} + 0.1 \cdot a^{n-m} \cdot (1-a)^m}$$

$$= \frac{1}{1 + 1/9 \cdot \left(\frac{1-a}{a}\right)^{2m-n}}$$

m	$P(X=0 \mid Z=m)$	
0	1.0000	
1	0.9998	$X=0$ most likely
2	0.9878	
3	0.5000	
4	0.0122	$X=1$ most likely
5	0.0002	

e)

m	LLR (for $P[X=0]=0.5$)	LLR (for $P[X=0]=0.9$
0	10.9861	13.1833
1	6.5917	8.7889
2	2.1972	4.3944
3	−2.1972	0.0000
4	−6.5917	−4.3944
5	−10.9861	−8.7899

Note: • both LLRs are linearly decreasing
- "slopes" are identical
- prior ($Pr[X=0]$) affects the offset.

Prob. 5.35:

a) • Y_1 and Y_2 are jointly Gaussian.

• Need to find:
- mean vector m
- covariance matrix C

<u>Mean vector:</u> $\underline{m} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} E[Y_1] \\ E[Y_2] \end{pmatrix}$

$$E[Y_1] = E\left[\int_0^2 (s(t) + N(t))\, dt\right]$$

$$= \int_0^2 \left(s(t) + \cancelto{=0}{E[N(t)]}\right) dt = 2$$

Similarly $E[Y_2] = 2$

<u>Covariance Matrix:</u> $C = \begin{pmatrix} Var(Y_1) & Cov(Y_1, Y_2) \\ Cov(Y_1, Y_2) & Var(Y_2) \end{pmatrix}$

$$Var(Y_1) = E\left[\left(\int_0^2 (s(t) + N(t))\, dt - \int_0^2 s(t)\, dt\right)^2\right]$$

$$= E\left[\left(\int_0^2 N(t)\, dt\right)^2\right] = \int_0^2 \int_0^2 \frac{N_0}{2} \delta(t-u)\, dt\, du$$

$$= 2 \cdot \frac{N_0}{2} = N_0$$

Similarly: $Var(Y_2) = N_0$

$$Cov(Y_1, Y_2) = E\left[\int_0^2 (s(t) + N(t))\, dt \cdot \int_1^3 (s(u) + N(u))\, du - \int_0^2 s(t)\, dt \cdot \int_1^3 s(u)\, du\right]$$

$$= E\left[\int_0^2 N(t)\, dt \int_1^3 N(u)\, du\right]$$

$$= \int_0^2 \int_1^3 \frac{N_0}{2} \cdot \delta(t-u)\, dt\, du$$

$$= \int_1^2 \frac{N_0}{2}\, dt = \frac{N_0}{2}$$

$$\Rightarrow \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \sim N\left(\begin{pmatrix} 2 \\ 2 \end{pmatrix}, \frac{N_0}{2}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\right) = N\left(\begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1/2 & 1/4 \\ 1/4 & 1/2 \end{pmatrix}\right)$$

b) Want $\Pr[Y_1 + Y_2 < 2]$.

Find pdf of $Y = Y_1 + Y_2$ first:

- Y is Gaussian
- $E[Y] = E[Y_1 + Y_2] = E[Y_1] + E[Y_2] = \underline{\underline{4}}$
- $\operatorname{Var}(Y) = E\Big[\big((Y_1 - \underbrace{2}_{E[Y_1]}) + (Y_2 - \underbrace{2}_{E[Y_2]})\big)^2\Big]$

$$= E\Big[(Y_1 - E[Y_1])^2 + 2(Y_1 - E[Y_1])(Y_2 - E[Y_2]) + (Y_2 - E[Y_2])^2\Big]$$

$$= \operatorname{Var}(Y_1) + 2 \cdot \operatorname{Cov}(Y_1, Y_2) + \operatorname{Var}(Y_2)$$

$$= N_0 + 2 \cdot \frac{N_0}{2} + N_0 = \underline{\underline{3N_0}} = \underline{\underline{3/2}}$$

$$\Pr[Y_1 + Y_2 < 2] = \int_{-\infty}^{2} \frac{1}{\sqrt{2\pi 3N_0}} \cdot \exp\left(-\frac{1}{2}\frac{(x-4)^2}{3N_0}\right) dx$$

$z = \frac{x-4}{\sqrt{3N_0}}$
$$= \int_{-\infty}^{\frac{2-4}{\sqrt{3N_0}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \, dz$$

$y = -z$
$$= \int_{\frac{4-2}{\sqrt{3N_0}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \, dy = Q\left(\frac{2}{\sqrt{3N_0}}\right) = Q\left(2\sqrt{\frac{2}{3}}\right)$$