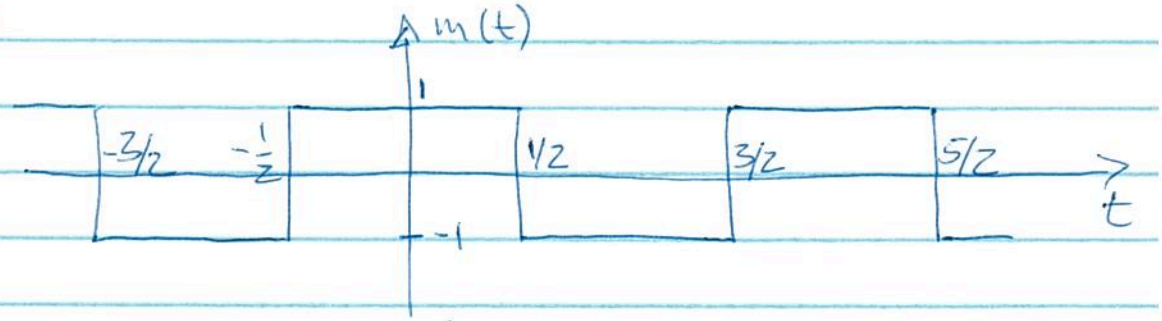
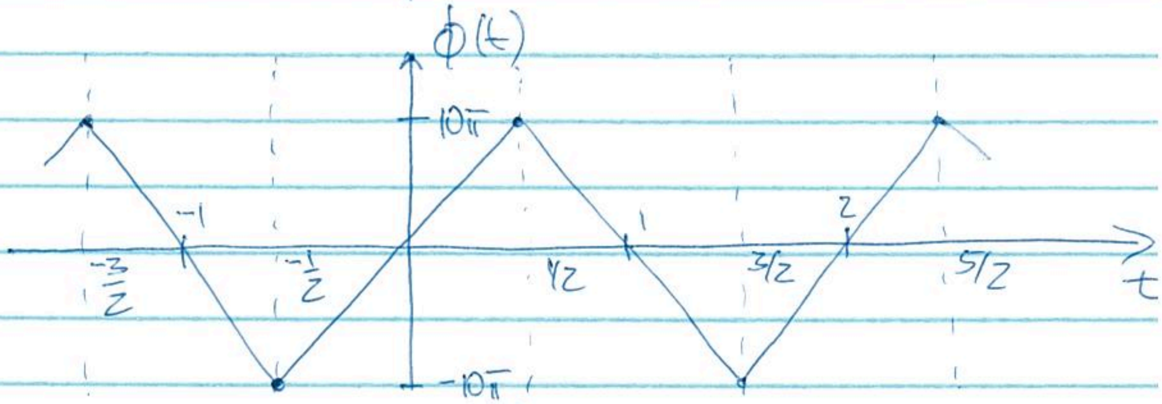
Dr. Paris

HW7 Solution

ECE 460

3.16 a)





b) Frequency deviation constant:

$$2\pi \cdot k_f = 20 \cdot \pi \Rightarrow k_f = 10$$

Carson's rule:

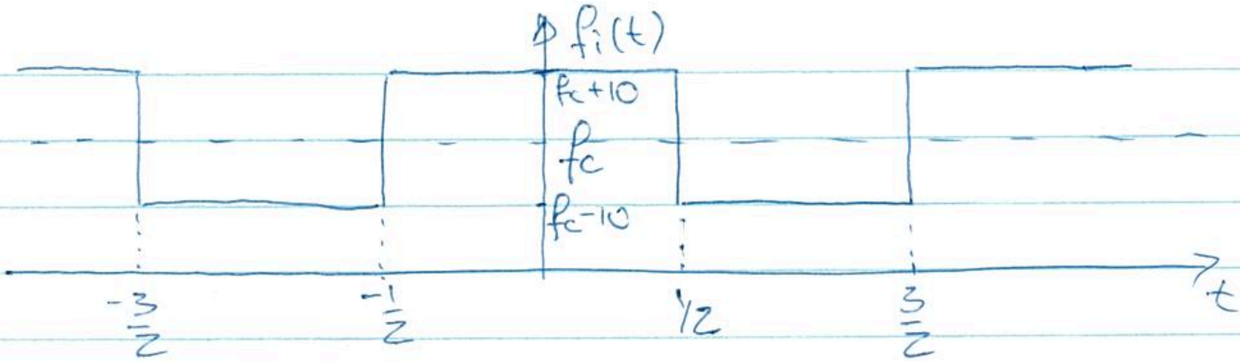
$$B_{FM} \approx 2 \cdot W + 2 \cdot \Delta f_{max}$$
$$= 2 \cdot 2 + 2 \cdot k_f \cdot \max|m(t)|$$
$$= 2 \cdot 2 + 2 \cdot 10 \cdot 1$$
$$= 24$$

c) Consider the instantaneous frequency of this signal:

$$f_i(t) = \frac{1}{2\pi} \cdot \frac{d\,\Psi(t)}{dt} \quad \Big| \; \Psi(t) = 2\pi f_c t + \phi(t)$$

$$= f_c + k_f \cdot m(t)$$

The question asks if the signal contains energy at frequencies $f_c + \alpha$, $\alpha \in \{0.5, 0.75, 1$



Since the instanteneous frequency hops back and forth between f_c-10 and f_c+10, one can argue that the signal does not contain energy at these 3 frequencies.

A more careful analysis proceeds as follows:
The complex envelope of $s(t)$ is

$$S_b(t) = 20 \cdot e^{j\phi(t)} \quad \text{(ref. frequency } f_c\text{)}$$

The plot in part (a) shows that $\phi(t)$ (and m(t
is periodic with period $T_0 = 2$. (fund. freq. $f_0 = \frac{1}{2}$)
Therefore, the signal $s_b(t)$ can be written as a Fourier series:

$$S_b(t) = \sum_n X_n \cdot e^{j2\pi n \cdot f_0 t}$$

Thus, $s_b(t)$ has a discrete spectrum and contains energy only at multiples of $f_0 = 0.5$.

$\Rightarrow$ $s(t)$ cannot contain energy at $f_c + 0.75$ since 0.75 is not a multiple of 0.5

$$X_n = \frac{1}{2}\int_{-1/2}^{3/2} S_b(t) \cdot e^{-j2\pi n f_0 t}\, dt$$

$$= \frac{1}{2}\int_{-1/2}^{1/2} 20 \cdot e^{j2\pi 10 t} \cdot e^{-j2\pi n f_0 t}\, dt + \frac{1}{2}\int_{1/2}^{3/2} 20 \cdot e^{-j2\pi 10 t} \cdot e^{-j2\pi n f_0 t}\, dt$$

$$= 10 \cdot \frac{\sin(\pi \cdot n \cdot f_0)}{\pi \cdot (10 - nf_0)} + 10 \cdot e^{-j\pi n} \cdot \frac{\sin(\pi n f_0)}{\pi \cdot (10 + nf_0)}$$

$$= \begin{cases} 10 & \text{for } n = \pm 20 \quad (\text{freq.} \pm 20 \cdot f_0 = \pm 10) \\ 0 & \text{for } n \text{ even}, n \neq \pm 20 \\ \frac{-200}{100 - (nf_0)^2} \cdot \sin(\frac{\pi}{2} n) & \text{for } n \text{ odd} \end{cases}$$

$\Rightarrow$ no energy at $\alpha = 2 \cdot f_0 = 1$

$\Rightarrow$ some energy $|X_1|^2 = \left(\frac{2}{1 - \left(\frac{1 \cdot f_0}{10}\right)^2}\right)^2 \approx 4$ at $\alpha = 1 \cdot f_0 = \frac{1}{2}$

As with the analysis based on inst. frequency the majority of energy is at $\pm 20 \cdot f_0$: $|X_{\pm 20}|^2 = 100$

4.1 a) Fourier transform of $p(t)$:

$$P(f) = \int_{-\infty}^{\infty} p(t) \cdot e^{-j2\pi ft} \, dt$$

$$= \int_0^1 \sin(\pi t) \cdot e^{-j2\pi ft} \, dt$$

$$= \frac{1}{2j} \cdot \int_0^1 \left(e^{j\pi t} - e^{-j\pi t}\right) \cdot e^{-j2\pi ft} \, dt$$

$$= \frac{1}{2j} \left[\int_0^1 e^{-j\pi(2f-1)t} \, dt - \int_0^1 e^{-j\pi(2f+1)t} \, dt \right]$$

$$= \frac{1}{2j} \cdot \left[\frac{e^{-j\pi(2f-1)} - 1}{-j\pi(2f-1)} - \frac{e^{-j\pi(2f+1)} - 1}{-j\pi(2f+1)} \right]$$

$$= \frac{1}{2\pi} \cdot \left[e^{-j\pi f} \cdot \frac{(\overbrace{e^{j\pi}}^{=-1} \cdot e^{-j\pi f} - e^{j\pi f})}{2f-1} - e^{-j\pi f} \cdot \frac{(\overbrace{e^{-j\pi}}^{=-1} \cdot e^{-j\pi f} - e^{j\pi f})}{2f+1} \right]$$

$$= \frac{e^{-j\pi f}}{\pi} \cdot \cos(\pi f) \cdot \left(\frac{-1}{2f-1} - \frac{-1}{2f+1} \right)$$

$$= \frac{e^{-j\pi f} \cdot \cos(\pi f)}{\pi} \cdot \frac{2}{1-(2f)^2}$$

$$= \frac{2\cos(\pi f)}{\pi \cdot (1-4f^2)} \cdot e^{-j\pi f}$$

b) Finding the $\gamma = 95\%$ containment BW is best solved numerically.

Define:

$$S_R(f) = \frac{\int_{-\infty}^{f} |P(v)|^2 dv}{\int_{-\infty}^{\infty} |P(v)|^2 dv}$$

then, the γ-containment BW (B_γ) can be found from $S_R(f)$ by recognizing that B_γ must satisfy:

$$S_R\left(\frac{B_\gamma}{2}\right) - S_R\left(-\frac{B_\gamma}{2}\right) = \gamma$$

Since $|P(f)|^2$ is even symmetric, $S_R(-\frac{B_\gamma}{2}) = 1 - S_R(\frac{B_\gamma}{2})$. Therefore,

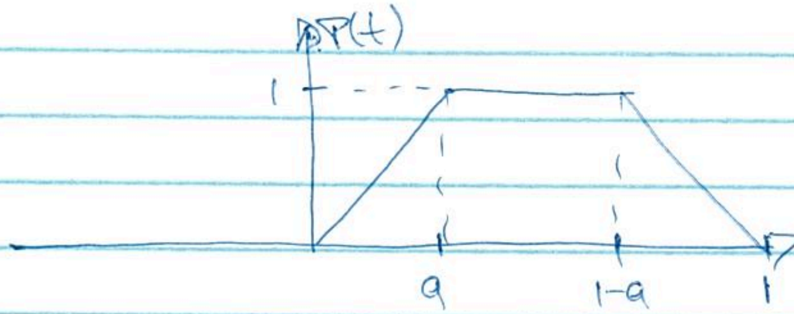
$$S_R\left(\frac{B_\gamma}{2}\right) \stackrel{!}{=} \frac{1+\gamma}{2}$$

$S_R(f)$ can be approximated in MATLAB:

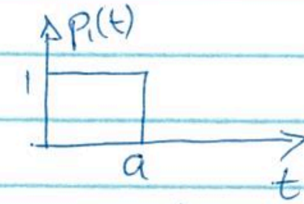
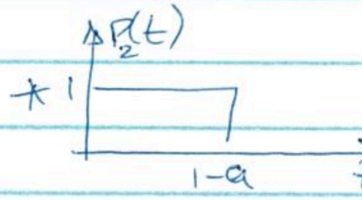
```
% find 95% containment BW of half-sine pulse
gamma = 0.95;
ff = -20:0.01:20;
P_num = 2*cos(pi*ff);
P_denom = pi*(1-4*ff.^2);

% at ff = +/-0.5, we need to use L'Hospitals rule, P(+/-0.5) = 0.5;
troubleInd = find(abs(abs(ff)-0.5) < sqrt(eps));
P_num(troubleInd) = 0.5;
P_denom(troubleInd) = 1;

% it's safe to divide now
PP = P_num./P_denom;

Sr = cumsum(abs(PP).^2)/sum(abs(PP).^2);

BgInd = find(Sr >= (1+gamma)/2, 1);
Bgamma = 2* ff(BgInd)
```

This yields $B_{95\%} = 1.82$.

4.2



The Fourier transform is best found, by recognizing that:

$p(t) = \frac{1}{a} \cdot$  $*$ 

The two rectangular pulses have Fourier transforms:

$$P_1(f) = a \cdot \text{sinc}(af) \cdot e^{-j2\pi f \cdot a/2}$$

and $$P_2(f) = (1-a) \cdot \text{sinc}((1-a)f) \cdot e^{-j2\pi f(1-a)/2}$$

Therefore:

$$P(f) = \frac{1}{a} \cdot P_1(f) \cdot P_2(f)$$

$$= (1-a) \cdot \text{sinc}(af) \cdot \text{sinc}((1-a)f) \cdot e^{-j\pi f}$$

b) For 4-PAM, average symbol power

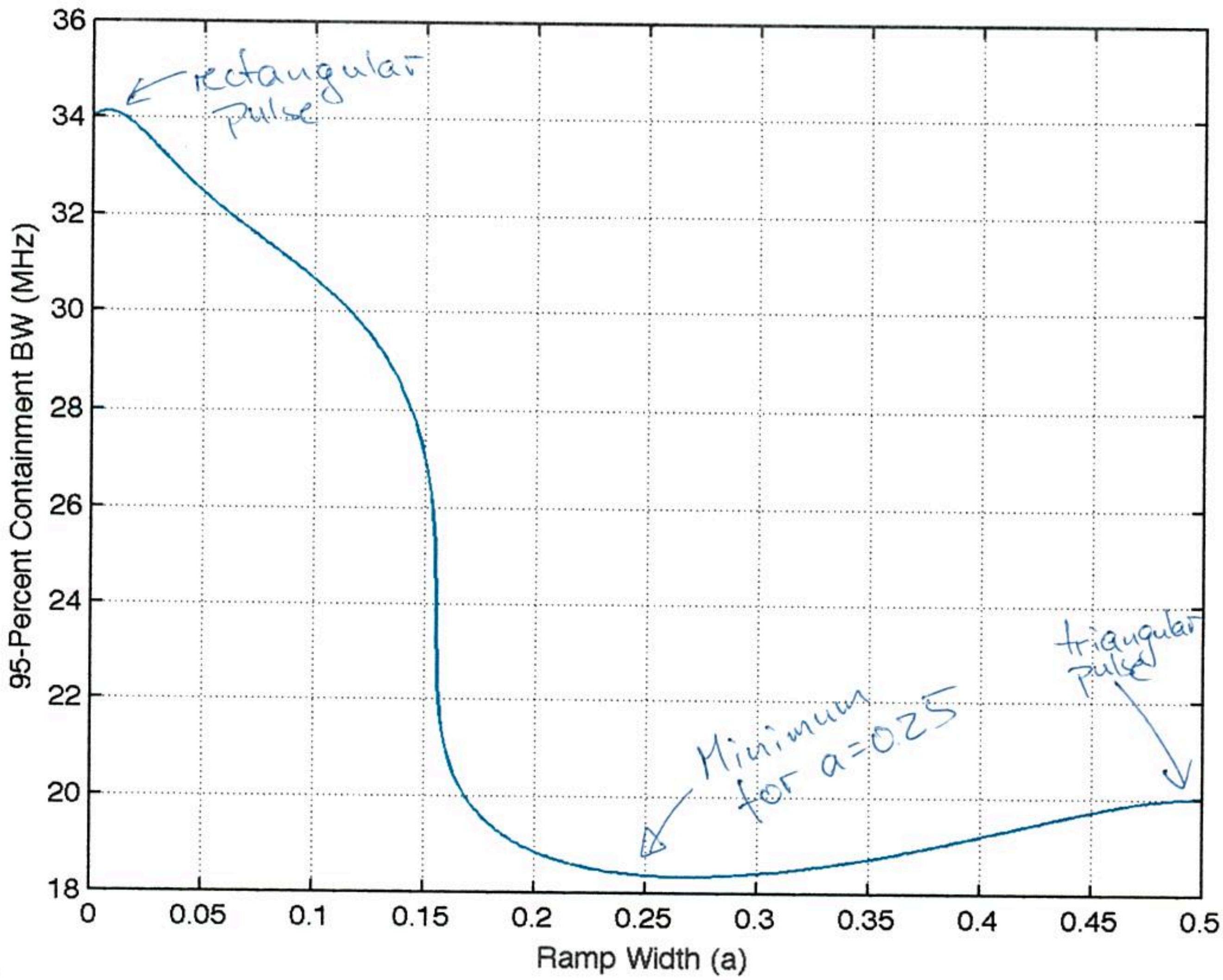
$$\sigma_b^2 = \frac{1}{4} \cdot \left[1^2 + (-1)^2 + 3^2 + (-3)^2\right] = 5$$

$\Rightarrow$ PSD: $S_a(f) = \frac{5}{T} \cdot |P(f)|^2$

c) Problem is best solved numerically.

See Problem 4.1 b for how to estimate fractional containment BW.

```
% find 95% containment BW of trapezoidal pulse as a function of ramp width
gamma = 0.95;
ff = -20:0.001:20;

aa = (0:0.001:0.5);
Bgamma = zeros(size(aa));

for nn = 1:length(aa)
    this_aa = aa(nn);
    PP = (1-this_aa)*sinc(this_aa*ff).*sinc((1-this_aa)*ff);

    Sr = cumsum(abs(PP).^2)/sum(abs(PP).^2);

    BgInd = find(Sr >= (1+gamma)/2, 1);
    Bgamma(nn) = 2* ff(BgInd);
end

plot(aa,10*Bgamma); % symbol period is 100ps => 1/T = 10MHz
grid on
xlabel('Ramp Width (a)');
ylabel(sprintf('%d-Percent Containment BW (MHz)', 100*gamma));
```



4.15. a) Smallest distance between pairs of neighboring points:

$$d_{min} = 2$$

b) The points of the constellation fall into three categories with equal energies:

1.) points: $\{\pm 1 \pm j\}$:
- 4 points
- energy $1^2 + 1^2 = 2$

2.) points: $\{\pm 1 \pm 3j\} \cup \{\pm 3 \pm j\}$
- 8 points
- energy $1^2 + 3^2 = 10$

3.) points: $\{\pm 3 \pm 3j\}$
- 4 points
- energy: $3^2 + 3^2 = 18$

$\Rightarrow$ Average Energy (per Symbol) $\sigma_b^2 = \frac{4}{16} \cdot 2 + \frac{8}{16} \cdot 10 + \frac{4}{16} \cdot 18 = 10$

c) Since constellation size $M = 16$, there are $\log_2 M = 4$ bits per symbol

$\Rightarrow$ Average Energy per bit : $E_b = \frac{\sigma_b^2}{4} = \frac{5}{2}$

$\Rightarrow$ Energy efficieny: $\eta_P = \frac{d_{min}^2}{E_b} = \frac{4}{5/2} = \frac{8}{5}$