ECE 460
Dr. Paris

HW#4 - Solution

1. Problem 2:16.

a) The problem asks for the inner product of the given passband signals

$$\langle u_1(t), u_2(t)\rangle = \int u_1(t)\cdot u_2(t)\,dt$$

This is easiest to determine via complex envelopes:

Passband	complex envelope
$u_1(t) = I_{[0,1]}(t)\cos(100\pi t)$	$u_{1,B}(t) = I_{[0,1]}(t)$
$u_2(t) = I_{[0,1]}(t)\cdot\sin(100\pi t)$	$u_{2,B}(t) = -j\cdot I_{[0,1]}$

The desired inner product can be found via:

$$\langle u_1(t), u_2(t)\rangle = \frac{1}{2}\cdot\mathrm{Re}\{\langle u_{1,B}(t), u_{2,B}(t)\rangle\}$$

with $\langle u_{1,B}(t), u_{2,B}(t)\rangle = \int u_{1,B}(t)\cdot u_{2,B}^*(t)\,dt$

$$= j\cdot\int_0^1 1\cdot dt = j$$

$\Rightarrow$ inner product is zero (signals are orthogonal)

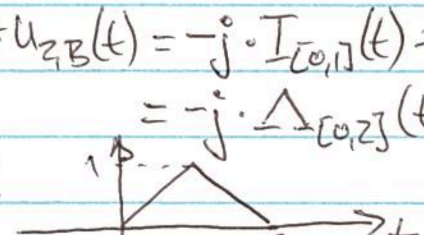
b) $u_1(t) * u_2(t)$ is best computed using complex envelopes

1.) Found complex envelopes in a)

2.) Need $\frac{1}{2}u_{1,B}(t) * u_{2,B}(t) = -j\cdot I_{[0,1]}(t) * I_{[0,1]}(t)$

$$= -j\cdot\Lambda_{[0,2]}(t)$$

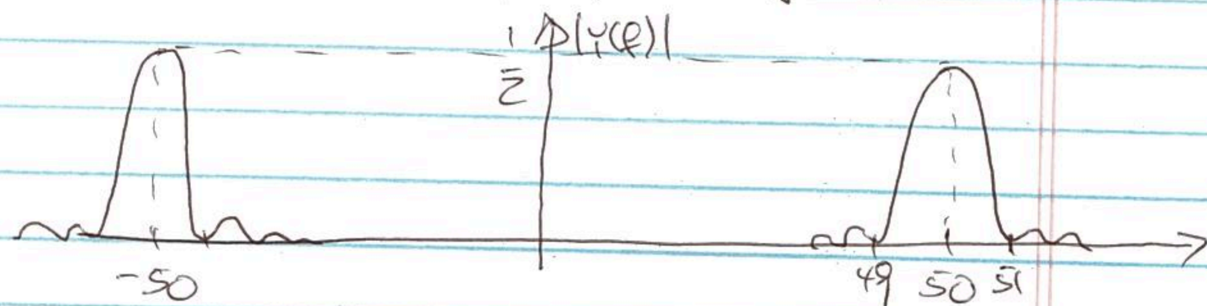
where $\Lambda_{[0,2]}(t)$:



3. convert to passband:

$$y(t) = u_1(t) * u_2(t) = \Lambda_{[0,2]}(t) \cdot \sin(100\pi t)$$

c) Magnitude of $Y(f)$: two $\text{sinc}^2(f)$ at $f=50$ and $f=-50$, respectively.



2.20

a) $u_p(t) = \text{sinc}(2t) \cdot \cos(100\pi t) \Rightarrow u(t) = \underline{\text{sinc}(2t)}$

$$v_p(t) = \text{sinc}(t) \cdot \sin(101\pi t + \tfrac{\pi}{4})$$

$$= \text{sinc}(t) \cdot \sin(100\pi t + \pi t + \tfrac{\pi}{4})$$

$\sin(a+b) = \sin(a)\cdot\cos(b) + \cos(a)\cdot\sin(b)$

$$= \text{sinc}(t) \cdot \left(\cos(100\pi t) \cdot \sin(\pi t + \tfrac{\pi}{4}) + \sin(100\pi t) \cdot \cos(\pi t + \pi/4) \right)$$

$$\Rightarrow v(t) = \text{sinc}(t) \cdot \left(\sin(\pi t + \tfrac{\pi}{4}) - j \cdot \cos(\pi t + \pi/4) \right)$$

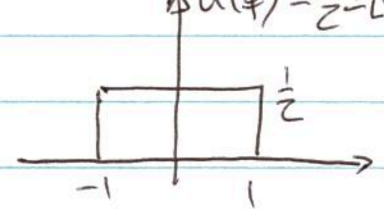
$\sin(a) = \cos(a - \frac{\pi}{2})$

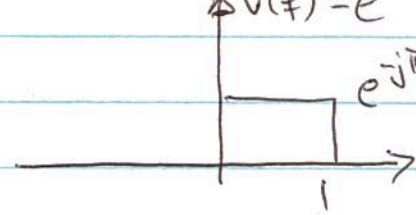
$\cos(a) = -\sin(a - \frac{\pi}{2})$

$$= \text{sinc}(t) \cdot \left(\cos(\pi t + \overbrace{\cancel{\tfrac{\pi}{4} - \tfrac{\pi}{2}}}^{-\pi/4}) + j \cdot \sin(\pi t + \overbrace{\cancel{\tfrac{\pi}{4} - \tfrac{\pi}{2}}}^{-\pi/4}) \right)$$

$$= \underline{\text{sinc}(t) \cdot e^{+j(\pi t - \pi/4)}}$$

b) $U(f) = \frac{1}{2} I_{[-1,1]}(f)$ $\qquad$ $V(f) = e^{-j\pi/4} \cdot I_{[0,1]}(f)$





$\Downarrow$

bandwidth of $u_p(t) = 2$ $\qquad$ bandwidt of $v_p(t) = 1$

<u>Note:</u> <u>two-sided</u> bandwidth of complex envelopes

c) 1.) $\langle u_p(t), v_p(t) \rangle = \frac{1}{2} \mathrm{Re}\{\langle u(t), v(t) \rangle\}$ $\quad (*)$

- This eliminates the carriers from the computation.
- However, sinc function are still tedious to integrate. $\Rightarrow$ transform to freq. domain. This is Parseval's relationship:

$$\langle u(t), v(t) \rangle = \int u(t) \cdot v^*(t)\, dt = \int U(f) \cdot V^*(f)\, df = \langle U(f), V(f) \rangle$$

$$\begin{aligned} \langle U(f), V(f) \rangle &= \int U(f) \cdot V^*(f)\, df \\ &= \int_{-\infty}^{\infty} \frac{1}{2} I_{[-1,1]}(f) \cdot e^{+j\pi/4} \cdot I_{[0,1]}(f)\, df \\ &= \frac{1}{2} e^{j\pi/4} \cdot \int_0^1 1 \cdot df = \frac{1}{2} e^{j\pi/4} \end{aligned}$$

Plugging back into $(*)$ above, it follows:

$$\begin{aligned} \langle u_p(t), v_p(t) \rangle &= \frac{1}{2} \mathrm{Re}\{\langle u(t), v(t) \rangle\} \\ &= \frac{1}{2} \cdot \mathrm{Re}\left\{\frac{1}{2} e^{j\pi/4}\right\} = \frac{1}{4} \cos(\pi/4) \\ &= \frac{1}{8} \sqrt{2} \end{aligned}$$

d) $y_p(t) = u_p(t) * v_p(t)$ is best solved with complex envelopes.

$\Rightarrow$ $y(t) = \frac{1}{2} u(t) * v(t)$

- This is still tedious; tranform to freq. domain

$$Y(f) = \frac{1}{2} \cdot U(f) \cdot V(f)$$

$$= \frac{1}{2} \cdot \frac{1}{2} I_{[-1,1]}(f) \cdot e^{-j\pi/4} \cdot I_{[0,1]}(f)$$

$$= \frac{1}{4} \cdot e^{-j\pi/4} \cdot I_{[0,1]}(f)$$

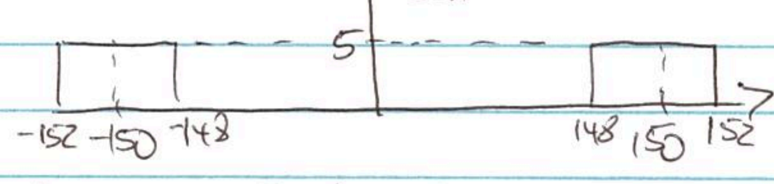
$\Rightarrow$ $y(t) = \frac{1}{4} \operatorname{sinc}(t) \cdot e^{j(\pi t - \pi/4)}$

$\Rightarrow$ $y_p(t) = \frac{1}{4} \cdot \operatorname{sinc}(t) \cdot \sin(101\pi t + \frac{\pi}{4})$

by comparison to $v_p(t)$ and $v(t)$, and $V(f)$

3.9 a) The spectrum is given by

$$U_{DSB}(f) = 5 \cdot \Pi(f-150) + 5 \cdot \Pi(f+150)$$

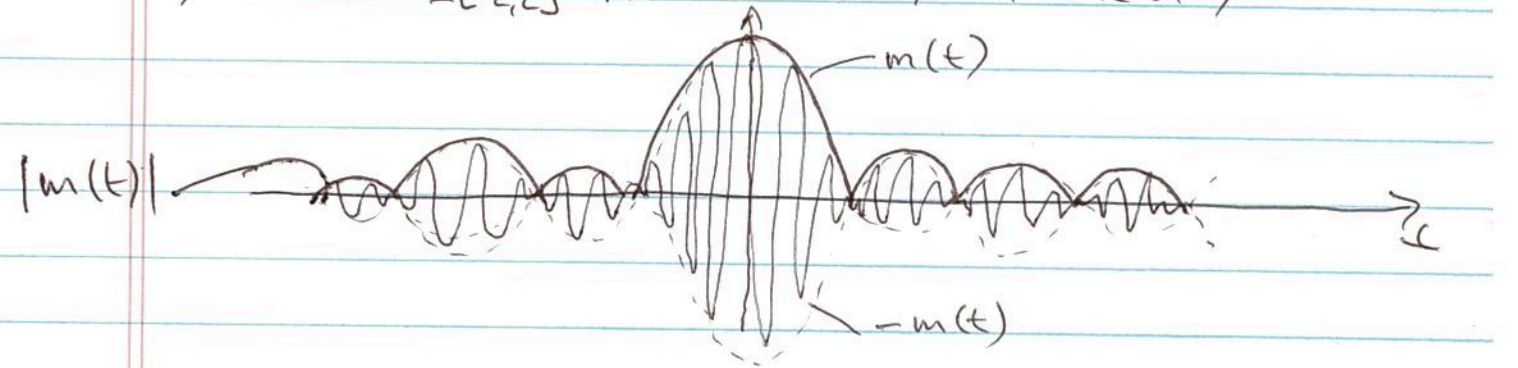


From figure: BW = 4

Energy: $\int u_{DSB}^2(t)\,dt \underset{\text{(Parseval)}}{=} \int U_{DSB}^2(f)\,df = 2 \cdot 5^2 \cdot 4 = 200$

Power: 0 (because energy is finite)

b) $M(f) = I_{[-2,2]}(f) \Longleftrightarrow m(t) = 4\,\text{sinc}(4t)$



Envelope detector finds $|m(t)|$

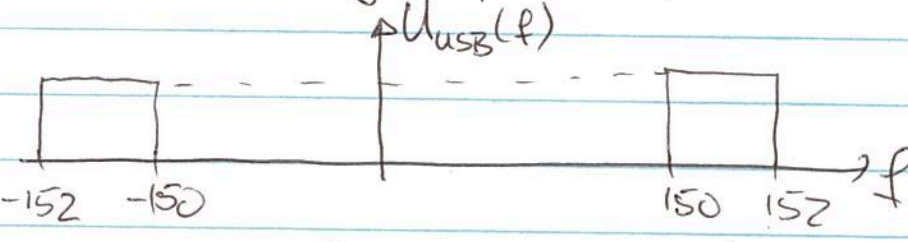
c) The minimum of the sinc(x) function occurs just before $x = 1.5$. The numerical minimum value is -0.2172.

we need $A_c + 4 \cdot \text{sinc}(4t) > 0$ for all t

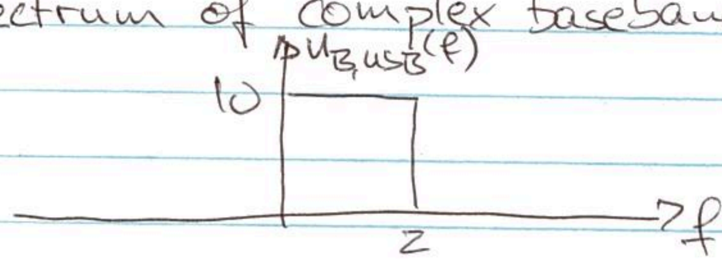
$$\Rightarrow A_c + 4 \cdot (-0.2172) > 0$$

$$\Rightarrow A_c \gtrsim 0.88$$

d) Spectrum after high pass filter:



$\Rightarrow$ Spectrum of complex baseband signal



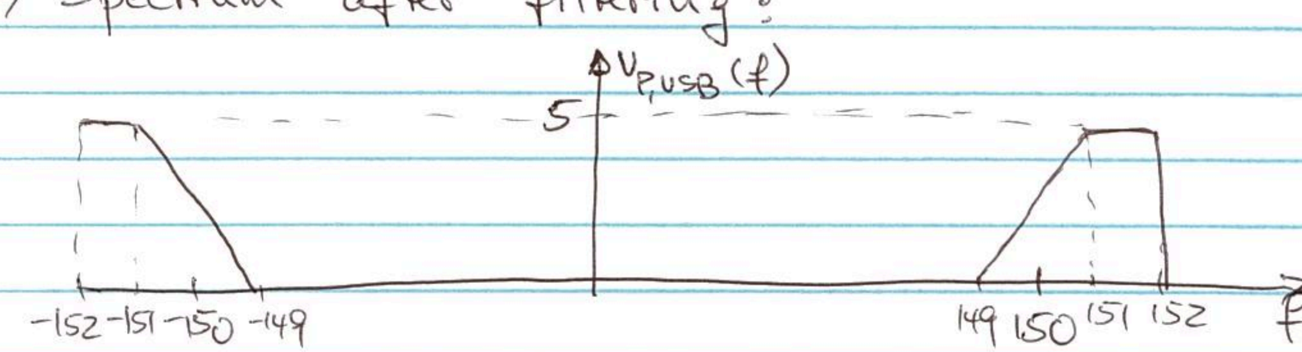
Note: reference frequency is $f = 150$.

Inv. Fourier transform of $U_{B,USB}(f)$ in figure above:

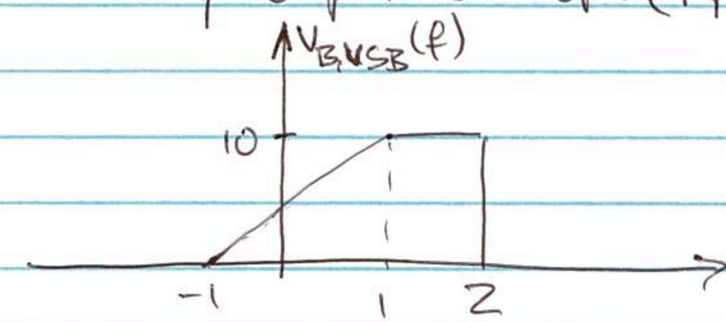
$$v_{B,USB}(t) = 20 \cdot \text{sinc}(2t) \cdot e^{j2\pi t}$$

(Inv. Fourier transform of $10 \cdot I_{[-1,1]}(f)$ shifted by 1 to the right.)

e) Spectrum after filtering:



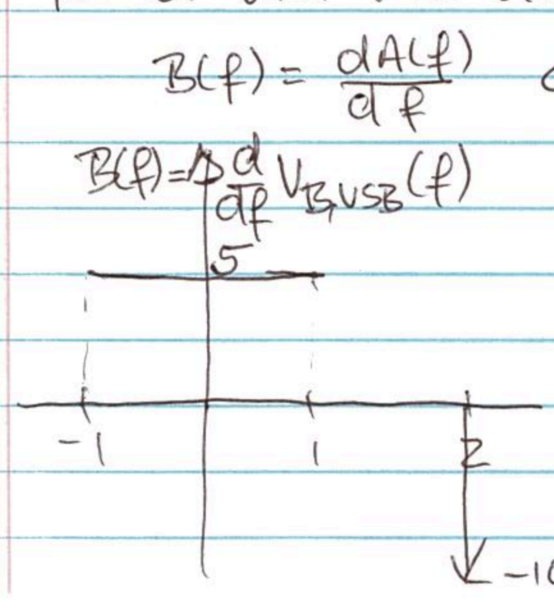
Spectrum of complex envelope (ref. frequency 150)



The inverse Fourier transform of $V_{B,USB}(f)$ is best found via the differention property

$$B(f) = \frac{dA(f)}{df} \longleftrightarrow b(t) = (-j2\pi t) \cdot a(t)$$

$B(f) = \frac{d}{df} V_{B,USB}(f)$



$$\longleftrightarrow b(t) = 10 \cdot \text{sinc}(2t) - 10 \cdot e^{j4\pi t}$$

$$\Rightarrow a(t) = v_{B,USB}(t) = \frac{b(t)}{-j2\pi t}$$

$$= \frac{10 \cdot \text{sinc}(2t) - 10 e^{j4\pi t}}{-j2\pi t}$$

$$\Rightarrow v_{B,VSB}(t) = j\cdot \frac{10\cdot \text{sinc}(2t)}{2\pi t} - j\cdot\frac{10(\cos(4\pi t)+j\sin(4\pi t))}{2\pi t}$$

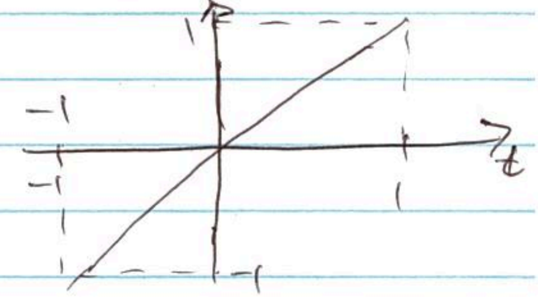
$$= 10\cdot\frac{\sin(4\pi t)}{2\pi t} + j\left(5\frac{\text{sinc}(2t)}{\pi t} - 5\frac{\cos(4\pi t)}{\pi t}\right)$$

$$= \underbrace{20\cdot \text{sinc}(4t)}_{v_c(t)} + j\cdot\underbrace{\left(\frac{5\,\text{sinc}(2t)}{\pi t} - \frac{5\cos(4\pi t)}{\pi t}\right)}_{v_s(t)}$$

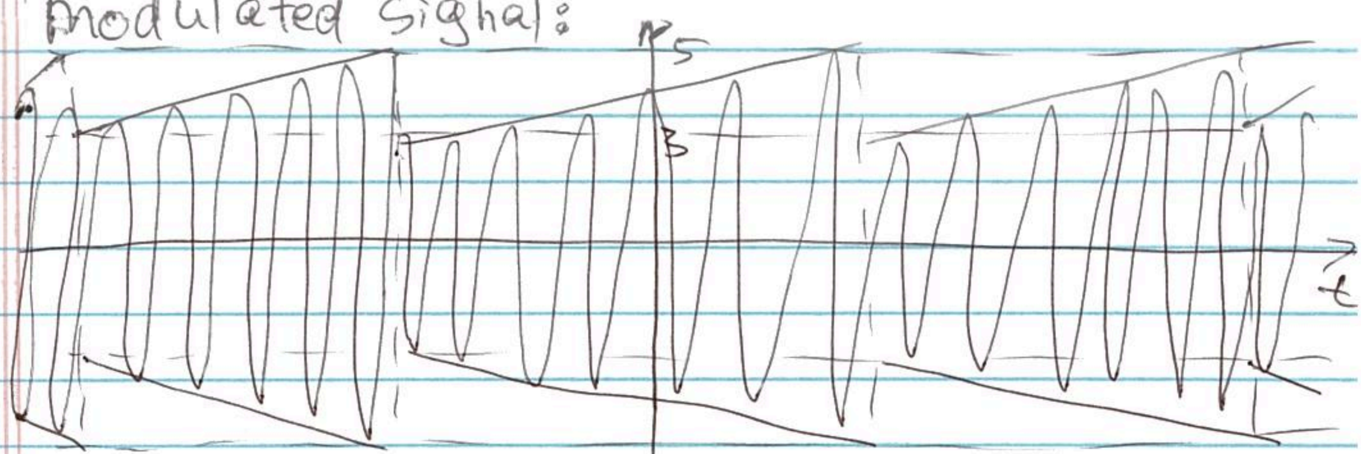
convert to passband:

$$v_{P,VSB}(t) = 20\cdot\text{sinc}(4t)\cdot\cos(2\pi 150 t) - \left(\frac{5\,\text{sinc}(2t)}{\pi t} - \frac{5\cos(4\pi t)}{\pi t}\right)\cdot\sin(2\pi 150 t)$$

3.11 $p(t) = t\cdot I_{[-1,1]}(t)$;



a) modulated signal:



b) power efficiency:

$$\eta_{AM} = \frac{a_{mod}^2\cdot\overline{m_n^2}}{1+a_{mod}^2\cdot\overline{m_n^2}} = \frac{1}{49} \approx 2\%$$

with: $a_{mod} = \frac{|\min(m(t))|}{A_c} = \frac{1}{4}$

$$\overline{m_n^2} = \frac{1}{2}\cdot\int_{-1}^{1}(p(t))^2 d = \frac{1}{2}\int_{-1}^{1} t^2 dt = \frac{1}{3}$$