ECE 460
Dr. Paris

HW #2 - Solution

2.9 a)

$$P(f) = \int_{-\infty}^{\infty} p(t) \cdot e^{-j2\pi f t}\, dt$$

$$= \int_{-\frac{1}{2}}^{1/2} \cos(\pi t) \cdot e^{-j2\pi f t}\, dt$$

$$= \int_{-\frac{1}{2}}^{1/2} \frac{1}{2}\left(e^{j\pi t} + e^{-j\pi t}\right) e^{-j2\pi f t}\, dt$$

$$= \frac{1}{2}\int_{-1/2}^{1/2} e^{-j(2\pi f - \pi)t}\, dt + \frac{1}{2}\int_{-\frac{1}{2}}^{1/2} e^{-j(2\pi f + \pi)t}\, dt$$

$$= \frac{1}{2} \cdot \left.\frac{e^{-j(2\pi f - \pi)t}}{-j(2\pi f - \pi)}\right|_{-\frac{1}{2}}^{1/2} + \frac{1}{2} \left.\frac{e^{-j(2\pi f + \pi)t}}{-j(2\pi f + \pi)}\right|_{-\frac{1}{2}}^{1/2}$$

$$= \frac{1}{2} \frac{e^{-j(\pi f - \pi/2)} - e^{j(\pi f - \pi/2)}}{-2j \cdot (\pi f - \pi/2)} + \frac{1}{2} \frac{e^{-j(\pi f + \pi/2)} - e^{j(\pi f + \pi/2)}}{-2j \cdot (\pi f + \pi/2)}$$

$\frac{e^{jx} - e^{-jx}}{2j} = \sin x$:

$$= \frac{\sin(\pi f - \pi/2)}{\pi \cdot (2f-1)} + \frac{\sin(\pi f + \pi/2)}{\pi \cdot (2f+1)}$$

$\sin(x - \frac{\pi}{2}) = -\cos x$
$\sin(x + \frac{\pi}{2}) = +\cos x$

$$= \frac{-\cos(\pi f)}{\pi \cdot (2f-1)} + \frac{\cos(\pi f)}{\pi \cdot (2f+1)}$$

$$= \frac{\cos(\pi f)}{\pi} \cdot \underbrace{\left(\frac{-1}{(2f-1)} + \frac{1}{(2f+1)} \right)}_{= \frac{-2}{4f^2-1}}$$

$$= \frac{2\cos(\pi f)}{\pi \cdot (1 - 4f^2)}$$

b) $u(t) = \sin(\pi t) \cdot I_{[0,1]}(t) = p(t - \frac{1}{2})$!

by delay property $\Rightarrow U(f) = P(f) \cdot e^{-j2\pi \cdot f \cdot \frac{1}{2}} = \frac{2\cos(\pi f)}{\pi(1-4f^2)} e^{-j\pi f}$

3.1 - This is a conventional (DSB) signal:

$$s(t) = \left(A_c + m(t)\right) \cdot \cos(2\pi f_c t)$$

- The upper envelope $(A_c + m(t))$ is a sinusoidal signal offset (shifted up) by $A_c = 20$

- The amplitude of $m(t)$ is $A_m = 10$; frequency and phase of $m(t)$ are $f_m = \frac{1}{0.5ms} = 2KHz$ and $\phi_m = 0$

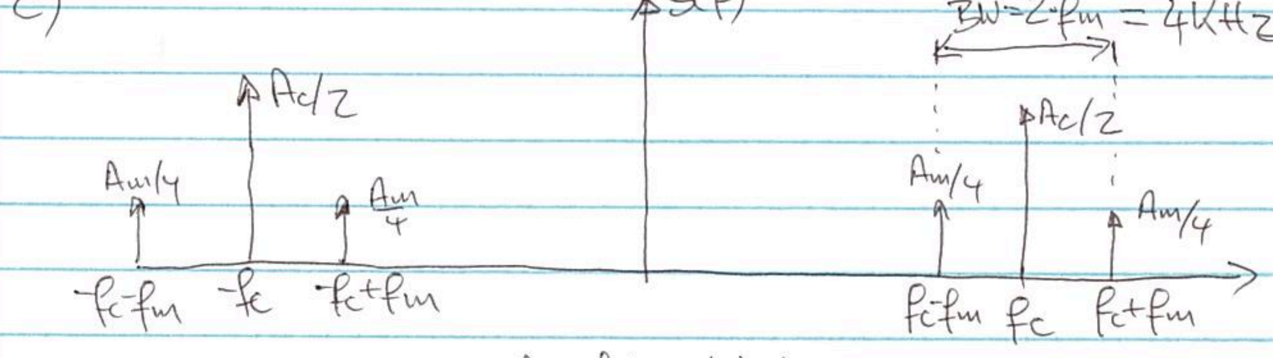
(a) Modulation index: $a_{mod} = \dfrac{|\min(m(t))|}{A_c} = \dfrac{10}{20} = 50\%$

(b) Signal power: begin by rewriting $s(t)$

$$s(t) = (A_c + m(t)) \cdot \cos(2\pi f_c t)$$
$$= A_c \cdot \cos(2\pi f_c t) + A_m \cdot \cos(2\pi f_m t) \cdot \cos(2\pi f_c t)$$
$$= A_c \cdot \cos(2\pi f_c t) + \frac{A_m}{2} \cdot \cos(2\pi (f_c - f_m) t) + \frac{A_m}{2} \cdot \cos(2\pi (f_c + f_m) t)$$

$$\text{Power of } s(t) = \frac{A_c^2}{2} + \frac{(A_m/2)^2}{2} + \frac{(A_m/2)^2}{2}$$
$$= \frac{A_c^2}{2} + \frac{A_m^2}{4} = 200 + 25 = 225$$

c)



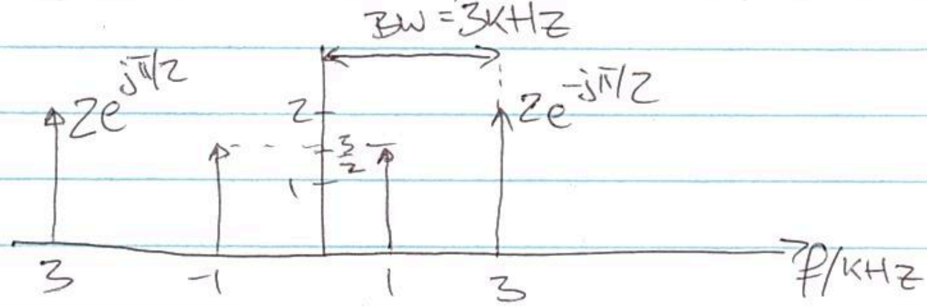
Bandwidth: 4KHz

3.3 a) $m(t) = 3 \cdot \cos(2\pi t) + 4 \cdot \sin(6\pi t)$

$$= 3 \cdot \cos(2\pi \cdot 1 \cdot t) + 4 \cdot \cos(2\pi \cdot 3 \cdot t - \pi/2)$$

$$\Updownarrow$$

$$M(f) = \frac{3}{2} \cdot \left(\delta(f-1) + \delta(f+1)\right) + 2 \cdot \left(e^{-j\pi/2}\delta(f-3) + e^{j\pi/2}\delta(f+3)\right)$$



Bandwidth: 3KHz

b) To find the normalized signal $m_n(t)$, we need $\min(m(t))$. This is difficult (impossible?) in closed form. To find minimum numerically, plot one period of $m(t)$ (e.g. in MATLAB)

The minimum value is $\min(m(t)) = -6.627$

```
tt=0:0.001:1;
mm=3*cos(2*pi*tt)
  +4*sin(6*pi*tt);
plot(tt,mm);
```

$$\Rightarrow M_0 = |\min(m(t))| = 6.627$$

$$\Rightarrow m_n(t) = \frac{m(t)}{M_0} = \frac{3}{6.627}\cos(2\pi t) + \frac{4}{6.627}\sin(2\pi t)$$

c) Given $a_{mod} = 50\% \Rightarrow 0.5 = \frac{M_0}{A_c}$

$$\Rightarrow A_c = 2 \cdot M_0 = 13.254$$

AM signal:

$$s(t) = (A_c + m(t)) \cdot \cos(2\pi f_c t)$$

$$= \left(13.254 + 3\cos(2\pi t) + 4\sin(2\pi 3t)\right) \cdot \cos(2\pi 600 t)$$

$$= A_c \cdot \left(1 + a_{mod} \cdot m_n(t)\right) \cdot \cos(2\pi f_c t)$$

d) Power efficiency : $\dfrac{\text{Power for sending message}}{\text{Total power}}$

Modulated message: $m(t) \cdot \cos(2\pi f_c t)$

$$= A_c \cdot a_{mod} \cdot m_n(t)$$

$$= A_c \cdot a_{mod} \cdot \left(\frac{3}{6.627}\cos(2\pi t) + \frac{4}{6.627}\sin(6\pi t)\right)$$

Power of this signal:

• power of $A \cdot \cos(2\pi f t + \phi)$: $\frac{A^2}{2}$

• power of sum of sinusoids = sum of powers of sinusoids

$$A_c^2 \cdot a_{mod}^2 \cdot \overbrace{\left(\frac{1}{2}\left(\frac{3}{6.627}\right)^2 + \frac{1}{2}\cdot\left(\frac{4}{6.627}\right)^2\right)}^{=\overline{m_n^2}}$$

$$= A_c^2 \cdot \frac{1}{4} \cdot \left(\frac{1}{2}\cdot 0.205 + \frac{1}{2}\cdot 0.364\right)$$

$$= A_c^2 \cdot 0.071$$

Total power :

$$A_c^2 \cdot \left(1 + a_{mod}^2 \cdot \overline{m_n^2}\right)$$

$$= A_c^2 \cdot (1 + 0.071) = A_c^2 \cdot 1.071$$

e) & f) on p. 6

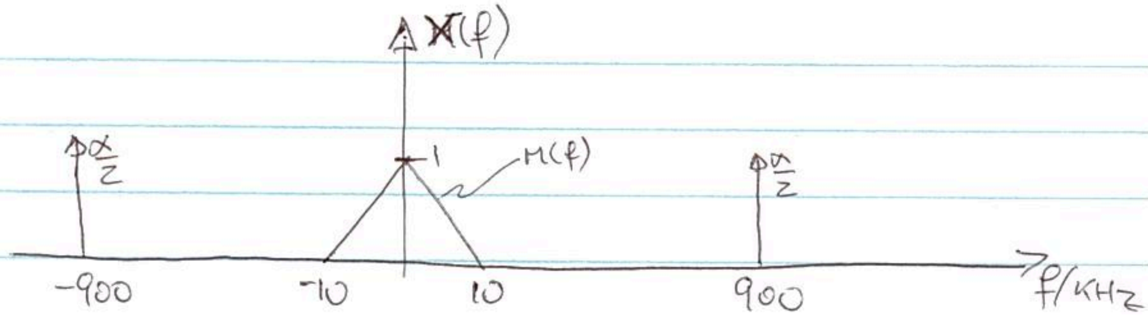
$$\Rightarrow \text{Efficiency} : \eta = \frac{\cancel{A_c^2} \cdot 0.071}{\cancel{A_c^2} \cdot 1.071} = 0.066 = 6.6\%$$

3.5 a) For this problem, we will look at the Fourièr transforms of $x(t)$ and $x^2(t)$ separately.

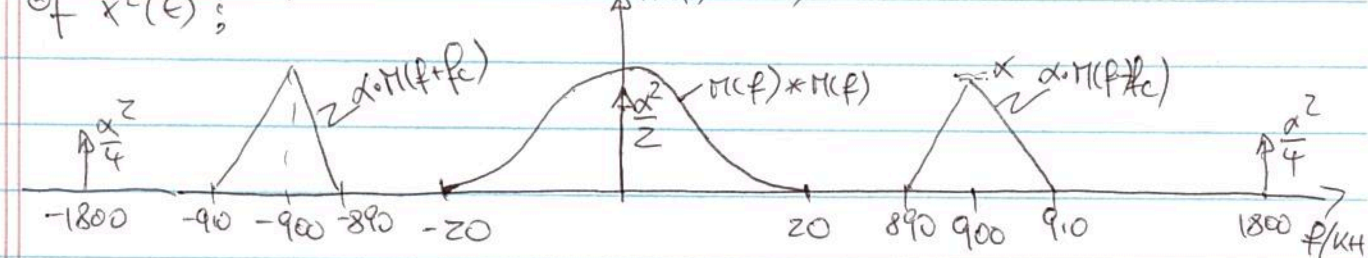
$$x(t) = m(t) + \alpha \cdot \cos(2\pi f_c t)$$

$$x^2(t) = \left(m(t) + \alpha \cdot \cos(2\pi f_c t)\right)^2$$

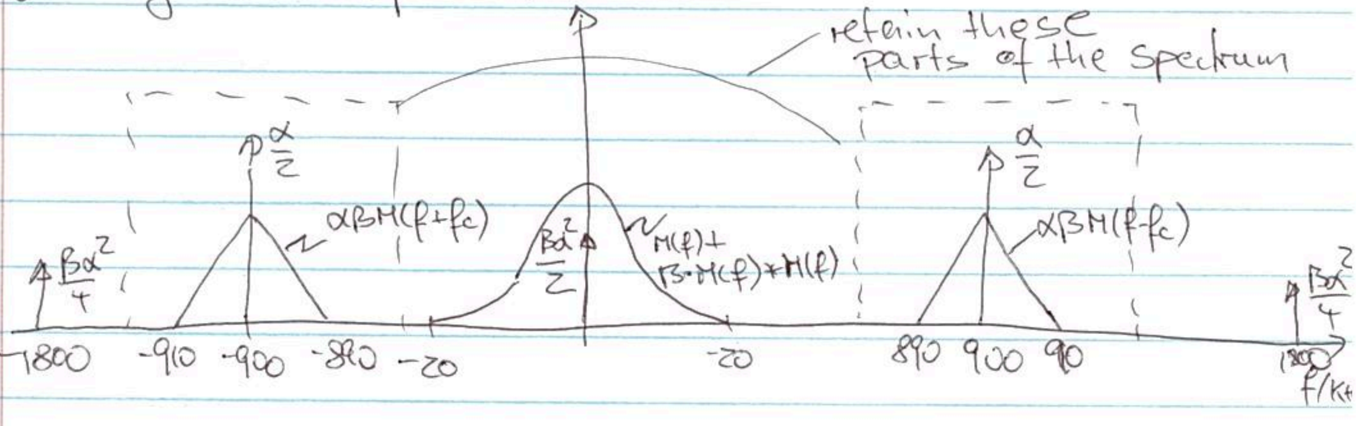
$$= m^2(t) + \alpha^2 \cdot \cos^2(2\pi f_c t) + 2\alpha \cdot m(t) \cdot \cos(2\pi f_c t)$$



Fourier transform of $x^2(t)$;



Fourier transform of $x(t) + \beta x^2(t)$ is obtained by adding above spectra:



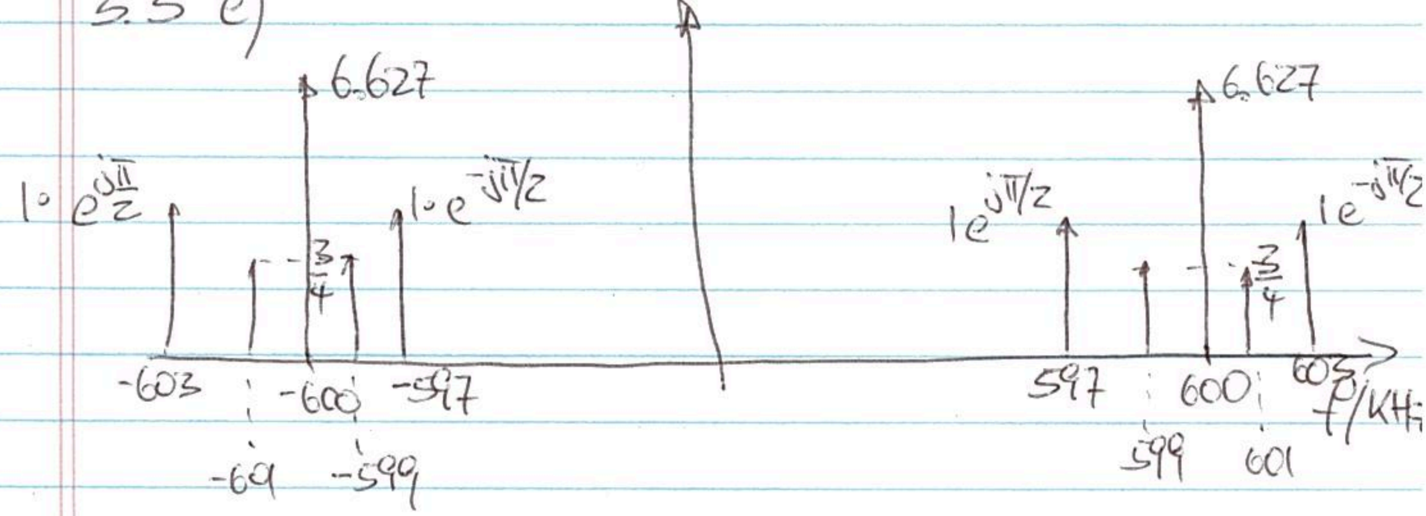
b) The bandpass filter must pass frequencies between 890 and 910 KHz:

Lower cut-off freq.: $20\,\text{KHz} \ll f_L \lesssim 890$

Upper cut-off freq.: $910 \lesssim f_u \ll 1800$

cont'd from p. 4

3.3 e)



f) The RC circuit is a low-pass filter. The time-constant $\tau = RC \approx \frac{1}{f_{cutoff}}$ must satisfy

$$\frac{1}{f_c} \ll RC \ll \frac{1}{B}$$

With the given values:

$$\frac{1}{600\,\text{KHz} \cdot 50\,\Omega} \ll C \ll \frac{1}{3\,\text{KHz} \cdot 50\,\Omega}$$

$$\Rightarrow \frac{1}{30\,\text{M}\Omega/\text{s}} \ll C \ll \frac{1}{150\,\text{K}\Omega/\text{s}}$$

$$\Rightarrow 33\,\text{nF} \ll C \ll 6\,\mu\text{F}$$

We would choose a cut-off frequency closer to $\frac{1}{B}$ than $\frac{1}{f_c}$, so $C = 500\,\text{nF}$ or $C = 1\,\mu\text{F}$ might be good choices.