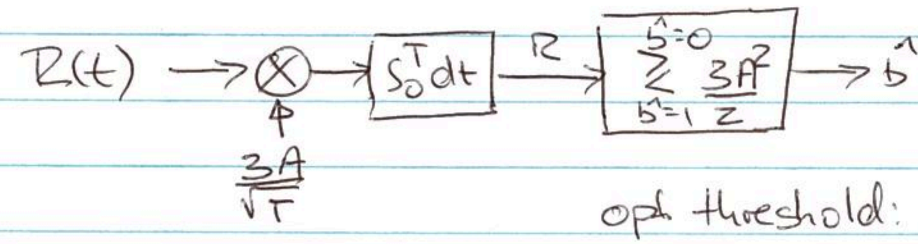
Dr. Paris

HW 10 - Solution

ECE 460

1.

a) opt. receiver

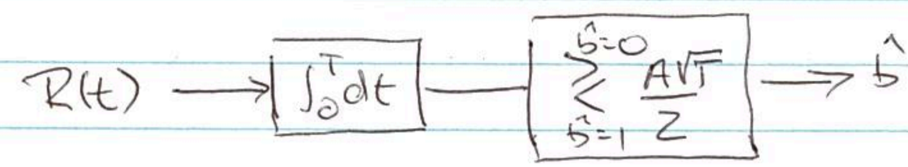


opt. threshold: $K = \frac{1}{2}(E_0 - E_1)$

$= \frac{1}{2}(4A^2 - A^2)$

$= \frac{3}{2}A^2$

alternatively, we can absorb the constant in the correlator into the theshold:



b) For the optimum receiver, the prob. of error is given by:

$$P_e = Q\left(\sqrt{\frac{E_0 + E_1 - 2r_{01}}{2N_0}}\right)$$

$E_0 = 4A^2$

$E_1 = A^2$

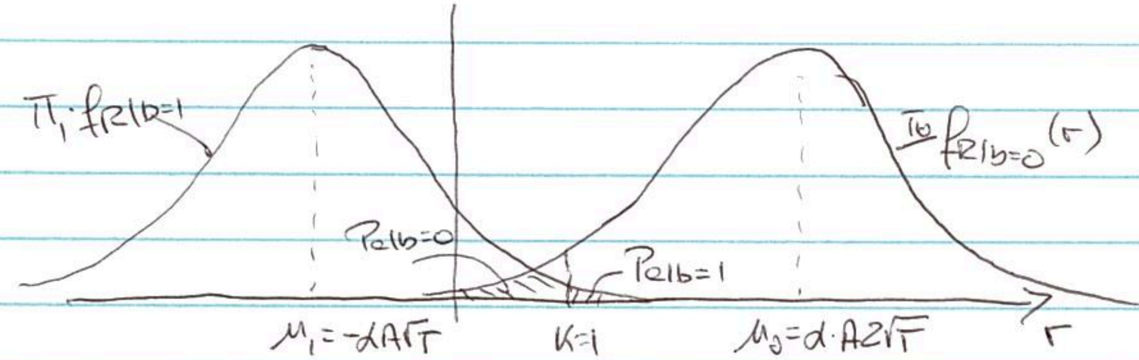
$r_{01} = 2A^2$

$$= Q\left(\frac{3A}{\sqrt{2N_0}}\right)$$

c) Conditioned on either $b=0$ or $b=1$, R will be Gaussian with the following means and variances:

	$b=0$	$b=1$
Mean: μ_0, μ_1	$\alpha \cdot 2A \cdot \sqrt{T}$	$-\alpha \cdot A \cdot \sqrt{T}$
Variance: σ^2	$\alpha^2 \cdot \frac{N_0}{2} \cdot T$	$\alpha^2 \cdot \frac{N_0}{2} \cdot T$

d)



For the fixed threshold ($K=1$), we cannot assume that conditional error probabilities $P_{e|b=0}$ and $P_{e|b=1}$ are equal => compute both

$$P_{e|b=1} = \int_1^{\infty} f_{R|b=1}(r|b=1)\,dr$$

$$= \int_1^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(r-\mu_1)^2}{2\sigma^2}\right) dr$$

$z = \frac{r-\mu_1}{\sigma}$:
$$= \int_{\frac{1-\mu_1}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = Q\left(\frac{1-\mu_1}{\sigma}\right) = Q\left(\frac{1+\alpha A\sqrt{T}}{\alpha \cdot \sqrt{N_0/2 \cdot T}}\right)$$

$$P_{e|b=0} = \int_{-\infty}^{1} \frac{1}{\sqrt{2\pi}\,\sigma} \cdot \exp\left(-\frac{(r-\mu_0)^2}{2}\right) dr$$

$z = -\frac{r-\mu_0}{\sigma}$:
$$\int_{\frac{\mu_0-1}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = Q\left(\frac{\mu_0-1}{\sigma}\right) = Q\left(\frac{\alpha 2A\sqrt{T}-1}{\alpha\sqrt{N_0/2T}}\right)$$

=> overall P_e:
$$P_e = \frac{1}{2} \cdot \left(Q\left(\frac{1+\alpha A\sqrt{T}}{\alpha\sqrt{N_0/2T}}\right) + Q\left(\frac{\alpha 2A\sqrt{T}-1}{\alpha\sqrt{N_0/2T}}\right) \right)$$

e) The gain α can be used to minimize P_e. For minimum P_e, the two conditional error probabilities must be equal. Equivalently, we can use α to ensure that the fixed threshold $K=1$ is optimal.

For equally likely bits the optimal threshold is the average of the means:

$$K_{opt} = \frac{1}{2}(\mu_0 + \mu_1) = \frac{1}{2}(\alpha 2A\sqrt{T} - \alpha A\sqrt{T})$$

$$= \alpha \cdot \frac{A\sqrt{T}}{2}$$

We want K_{opt} to equal the fixed $K=1$:

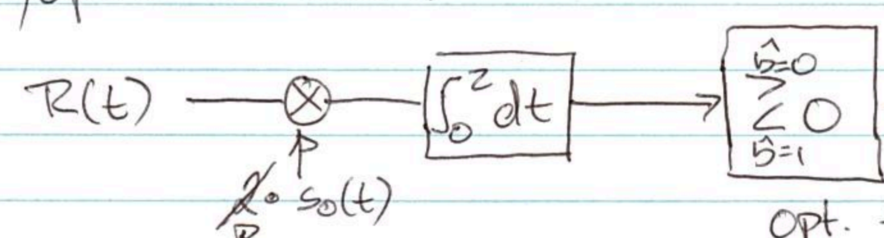
$$\Rightarrow 1 = \alpha \frac{A\sqrt{T}}{2}$$

$$\Rightarrow \alpha = \frac{2}{A\sqrt{T}}$$

Plugging this value of α into the conditional probabilities of error yields:

$$P_{e|b=0} = P_{e|b=1} = P_e = Q\left(\frac{3A}{\sqrt{2N_0}}\right)$$ (same as (b) $\Rightarrow$ opt. receiver)

2. a) opt. receiver:



not needed since threshold is 0.

Opt. threshold is 0 since $s_1(t) = -s_0(t)$ $\Rightarrow E_0 = E_1$.

b) $E_0 = E_1 = \int_0^2 s_0^2(t)\,dt$

$$= 4\cdot\int_0^{1/2}(2t)^2\,dt$$

$$= 4\cdot \frac{4t^3}{3}\Big|_0^{1/2} = \frac{16}{3}\cdot\left(\frac{1}{2}\right)^3 = \frac{2}{3}$$

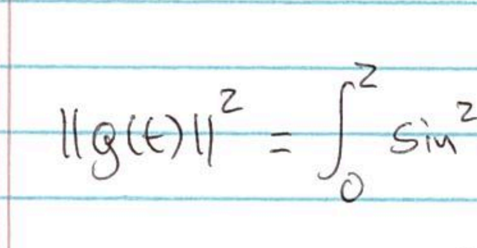
$$r_{01} = \int_0^2 s_0(t)\cdot s_1(t)\,dt \underset{s_1(t)=-s_0(t)}{\overset{\uparrow}{=}} -\int_0^2 s_0^2(t)\,dt = -E_0 = -\frac{2}{3}$$

$$\Rightarrow P_e = Q\left(\sqrt{\frac{E_0+E_1-2r_{01}}{2N_0}}\right) = Q\left(\sqrt{\frac{4\cdot 2/3}{2N_0}}\right) = Q\left(\sqrt{\frac{4}{3N_0}}\right)$$

c) For this receiver, the probability of error equals:

$$P_e = Q\left(\frac{\langle s_0(t)-s_1(t),\, g(t)\rangle}{\sqrt{2N_0\cdot \|g(t)\|^2}}\right)$$

$$\langle s_0(t)-s_1(t),\, g(t)\rangle = \int_0^2 2\cdot s_0(t)\cdot \sin(\pi t)\,dt$$



$$= 4\cdot\int_0^{1/2} 2\cdot 2t\cdot\sin(\pi t)\,dt$$

$$= 16\cdot\left(\frac{\sin(\pi t) - \pi t\cos(\pi t)}{\pi^2}\right)\Big|_0^{1/2}$$

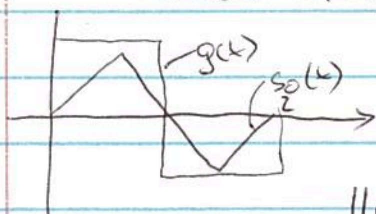
$$= \frac{16}{\pi^2}$$

$$\|g(t)\|^2 = \int_0^2 \sin^2(\pi t)\,dt = \int_0^2\left(\frac{1}{2}-\frac{1}{2}\cos(2\pi t)\right)dt = 1$$

$$\Rightarrow P_e = Q\left(\frac{16}{\pi^2\sqrt{2N_0}}\right)$$

d) Proceeding similarly:

$$\langle s_0(t) - s_1(t), g(t) \rangle = \int_0^2 2 \cdot s_0(t) \cdot g(t)\, dt$$



$$= 4 \cdot \int_0^{1/2} 2 \cdot 2t\, dt = \frac{16t^2}{2}\Big|_0^{1/2} = 2$$

$$\|g(t)\|^2 = \int_0^2 g^2(t)\, dt = 2 \cdot \int_0^1 1^2 dt = 2$$

$$\Rightarrow P_e = Q\left(\frac{\langle s_0(t) - s_1(t), g(t) \rangle}{\sqrt{2N_0 \cdot \|g(t)\|^2}} \right) = Q\left(\frac{2}{\sqrt{2N_0 \cdot 2}} \right)$$

$$= Q\left(\frac{1}{\sqrt{N_0}} \right)$$

e) Receiver (b) is best (d) is worst.

Consider:

$$\underbrace{\sqrt{\frac{4}{3}}}_{\approx 1.1547} \cdot \frac{1}{\sqrt{N_0}} > \underbrace{\frac{16}{\pi^2 \sqrt{2}}}_{1.1146} \cdot \frac{1}{\sqrt{N_0}} > 1 \cdot \frac{1}{\sqrt{N_0}}$$

- We know, that matching with $s_0(t) - s_1(t)$, i.e., triangular signals, is optimum.

- The sin signal in (c) is very similar to triangles ⇒ small loss in performance

- The rectangular signals in (d) are least like the triangles ⇒ worst of the three.

3. a)

$$R(t) \longrightarrow \otimes \longrightarrow \boxed{\int_0^T dt} \xrightarrow{\;R\;} \boxed{\underset{\hat{b}=1}{\overset{\hat{b}=0}{\gtrless}}\; 0} \longrightarrow \hat{b}$$

$\cancel{2}\cdot s_0(t)$

not needed since $K=0$

$K = \frac{1}{2}(E_0 - E_1) + \frac{N_0}{2}\ln\left(\frac{\pi_1}{\pi_0}\right) = 0$

Since $\pi_0 = \pi_1$ and $E_0 = E_1$

b) $$P_e = Q\left(\frac{\|s_0(t) - s_1(t)\|}{\sqrt{2N_0}}\right)$$

$$\|s_0(t) - s_1(t)\|^2 = \int_0^T (s_0(t) - s_1(t))^2\, dt$$

$$= \int_0^T 4\cdot s_0^2(t)\, dt$$

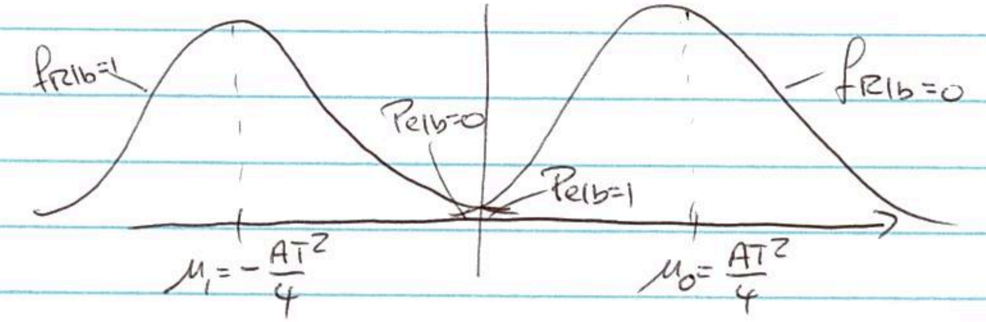
$$= 2\cdot\int_0^{T/2} 4\cdot (At)^2\, dt = \frac{8A^2t^3}{3}\Big|_0^{T/2} = \frac{A^2T^3}{3}$$

$$\Rightarrow P_e = Q\left(AT\cdot\sqrt{\frac{T}{6N_0}}\right)$$

c) conditional distributions of R are Gaussian:

	$b=0$	$b=1$
mean:	$\mu_0 = \langle s_0(t), g(t)\rangle = 2\cdot\int_0^{T/2} At\, dt = \frac{AT^2}{4}$	$\mu_1 = \langle s_1(t), g(t)\rangle = -\frac{AT^2}{4}$
Variance:	$\sigma^2 = \frac{N_0}{2}\|g(t)\|^2 = \frac{N_0}{2}\cdot T$	$\sigma^2 = \frac{N_0}{2}T$

d)

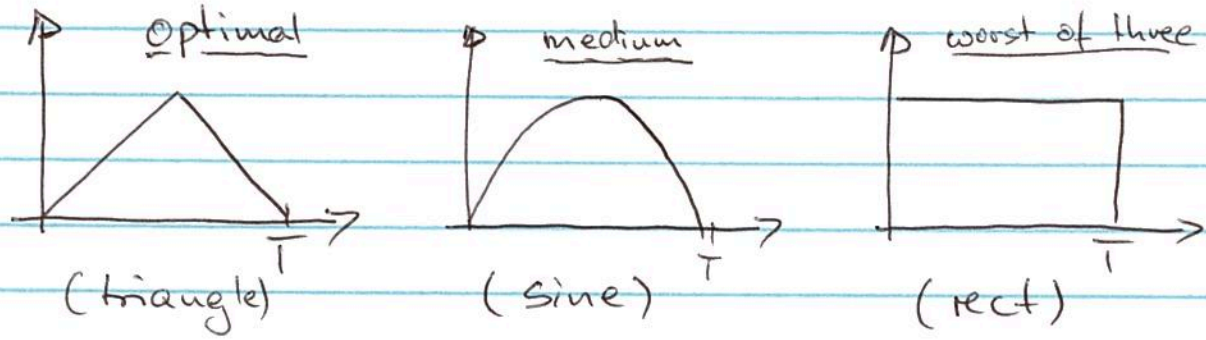


By symmetry, $P_{e|b=0} = P_{e|b=1} = P_e = Q\left(\frac{\mu_0}{\sigma}\right)$

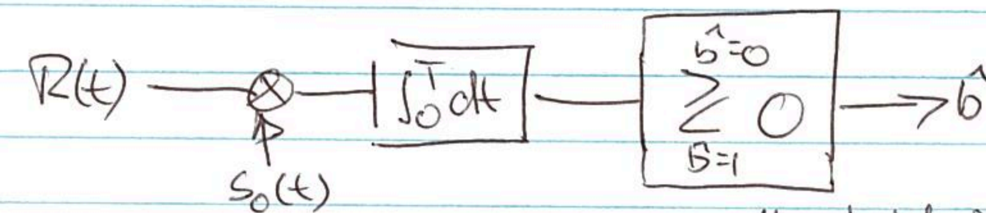
$$= Q\left(\frac{AT^2/4}{\sqrt{N_0/2 \cdot T}}\right)$$

$$= Q\left(AT \cdot \sqrt{\frac{T}{8N_0}}\right)$$

e) Sinusoidal pulses would give lower P_e than the rectangular pulses (but not as good as the optimal triangular pulses). The sine pulse is a better approximation to the triangles than rectangular pulses:



4. a)



threshold is zero, since signals are equally likely and have equal energy.

b) $P_e = Q\left(\sqrt{\dfrac{E_0+E_1-2r_{01}}{2N_0}}\right)$

$= Q\left(A\cdot\sqrt{\dfrac{4T}{3\cdot 2N_0}}\right)$

$= Q\left(A\cdot\sqrt{\dfrac{2T}{3N_0}}\right)$

$E_0 = E_1 = -r_{01} = \int_0^T s_0^2(t)\,dt$

$= 2\cdot\int_0^{T/2}\left(\frac{2At}{T}\right)^2 dt$

$= 2\cdot\frac{4A^2}{T^2}\,\frac{t^3}{3}\Big|_0^{T/2}$

$= \dfrac{A^2T}{3}$

c) conditioned on b=0 and b=1, respectively, R is Gaussian with following means & variances:

	b=0	b=1
Mean:	$\mu_0 = \int_0^T s_0(t)\,dt$ $= 2\cdot\int_0^{T/2}\left(\frac{2At}{T}\right)dt$ $= \frac{AT}{2}$	$\mu_1 = \int_0^T s_1(t)\,dt$ $= -\frac{AT}{2}$
Variance:	$\sigma^2 = \frac{N_0}{2}\int_0^T 1\,dt$ $= \frac{N_0}{2}\cdot T$	$\sigma^2 = \frac{N_0}{2}T$

d) By symmetry: $P_e = P_{e|b=0} = P_{e|b=0}$

$$= Q\left(\frac{\mu_0}{\sigma}\right)$$

$$= Q\left(\frac{AT/2}{\sqrt{N_0/2\ T}}\right) = Q\left(A \cdot \sqrt{\frac{T}{2N_0}}\right)$$

e) When the transmitted signal is amplified by α, then:

- Means in part (c) change to
$$\mu_0 = \alpha \frac{AT}{2} \quad \text{and} \quad \mu_1 = -\alpha \frac{AT}{2}$$
- Variance is unchanged
- Probability of error changes to:
$$P_e = Q\left(\alpha A \cdot \sqrt{\frac{T}{2N_0}}\right)$$

To make this P_e equal to the optimal P_e (in part (b)):

$$\alpha \cdot A \cdot \sqrt{\frac{T}{2N_0}} = A \cdot \sqrt{\frac{2T}{3N_0}}$$

$$\Rightarrow \quad \alpha = \sqrt{\frac{4}{3}}$$

f) Part (e) says that with the suboptimum receiver, we must increase transmitted power by $\alpha^2 = \frac{4}{3}$ to achieve P_e of optimum receiver. Thus, the suboptimum loses $\alpha^2 = \frac{4}{3} = 1.25\text{dB}$ in relation to the optimum receiver.