ECE 732: Mobile Communication Systems
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Solution to Homework 1

1. Problem 1

   (a) Inner product:

$$(s_1(t), s_2(t)) = \int_0^T A^2 \cos(2\pi(f_c + \frac{\Delta f}{2})t) \cdot \cos(2\pi(f_c - \frac{\Delta f}{2})t)dt.$$

   Using $\cos(a) \cdot \cos(b) = \frac{1}{2}(\cos(a+b) + \cos(a-b))$ the inner product becomes

$$(s_1(t), s_2(t)) = \frac{A^2}{2} \int_0^T (\cos(4\pi f_c t) + \cos(2\pi \Delta f t))dt.$$

   Now, the assumption $f_c \gg \Delta f$ implies that the integral over $\cos(4\pi f_c t)$ is negligibly small and the inner product equals

$$(s_1(t), s_2(t)) \approx \frac{A^2 T}{2} \text{sinc}(2\pi \Delta f T).$$

   Norms:

$$||s_1(t)||^2 \approx ||s_2(t)||^2 \approx \frac{A^2 T}{2}.$$

   using again the assumption $f_c \gg \Delta f$.
   In summary,

$$\rho = \frac{(s_1(t), s_2(t))}{||s_1(t)|| \cdot ||s_2(t)||} \approx \frac{\sin(2\pi \Delta f T)}{2\pi \Delta f T} = \text{sinc}(2\pi \Delta f T).$$

   (b) $\Delta f = \frac{1}{2T}$ is the smallest value for which signals are orthogonal.

   (c) Optimum (matched filter) receiver consists of the following three elements:

   - multiplication with $s_1(t) - s_2(t)$,
   - integration from 0 to T,
   - decision device: decide $s_1(t)$ was sent when output of integrator is positive; otherwise decide $s_2(t)$ was sent.

(d) For a matched filter receiver, the probability of error is given by

$$P_e = \mathrm{Q}(\frac{d}{\sqrt{2N_0}}),$$

where $d^2 = ||s_1(t) - s_2(t)||^2$. Given the above calculations, we find

$$d^2 = ||s_1(t)||^2 + ||s_s(t)||^2 - (s_1(t), s_2(t)) \approx A^2T(1-\rho)$$

and, thus,

$$P_e \approx = \mathrm{Q}(A\frac{T(1-\rho)}{2N_0}).$$

Clearly, P_e is minimized when ρ is minimized. Since ρ depends on Δf via a sinc-function, the smallest value of ρ corresponds to the minimum of the sinc between the first and second zero-crossings. The location of this minimum is approximately equal to $\Delta f = \frac{3}{4T}$ and the corresponding value is $\rho = -\frac{2}{3\pi}$.

2. Problem 2

(a) The optimum receiver is equivalent to an integrator (from 1 to 2) followed by a decision device. The decision device decides $s_0(t)$ was sent if output of integrator is positive.

The simplified form of the matched filter receiver follows from the observation that $s_0(t) = s_1(t)$ between 0 and 1.

(b) Using $P_e = \mathrm{Q}(\frac{d}{\sqrt{2N_0}})$, we easily find $P_e = \mathrm{Q}(\sqrt{\frac{2}{N_0}})$.

(c) Receiver and probability of error are identical for this signal set!

(d) The reason for identical receiver and performance with the two signal sets is that the signal difference $s_0(t) - s_1(t)$ is identical for both signal sets; matched filter and error performance depend only on signal difference.

Second signal set uses less energy per bit. First set is useful when phase of the received signal is unknown (differential phase shift keying).

3. Problem 3

(a) It is useful for the remainder of the problem to consider the following the receiver structure.

- First branch correlates with a sinusoid of frequency f_0, i.e., the received signal is multiplied with a sinusoid of frequency f_0 and then integrated from 0 to T.
- The second branch correlates the received signal with a sinusoid of frequency f_1.
- The difference of the two brances (first minus second) is fed into a decision device; this device decides the sinusoid of frequency f_0 was transmitted when the difference is positive.

(b) The assumptions about the signals imply that the signals are orthogonal and that each signal requires the same bit energy $E_b = A_0^2 T$. Consequently, the probability of error equals $P_e = \mathrm{Q}(\sqrt{\frac{A_0^2 T}{N_0}})$.

The requirement that $P_e = 10^{-3}$ implies that the bit duration T must satisfy:

$$T = \mathrm{Q}^{-1}(10^{-3}) \cdot \frac{N_0}{A_0^2}.$$

The data rate is the inverse of T.

(c) For the receiver described in the problem, the multiplication with the sinusoids is replaced with a multiplication with square waves of identical frequencies and phases. This implies the following statistics

- Mean of integrator in first branch (f_0), when a sinusoid of frequency f_0 was sent, is $\sqrt{2} A_0 \frac{T}{\pi}$. When a sinusoid of frequency f_1 was sent, the mean is 0.
- Mean of integrator in second branch (f_1), when a sinusoid of frequency f_1 was sent, is $\sqrt{2} A_0 \frac{T}{\pi}$. When a sinusoid of frequency f_0 was sent, the mean is 0.
- Variance of difference of the integrator ouputs is $N_0 T$.

From these results, it follows that the probability of error increases to:

$$P_e = \mathrm{Q}(\sqrt{\frac{A_0^2 T}{N_0}} \frac{\sqrt{2}}{\pi}).$$

(d) The receiver must know be converted to an energy detector. In each of the branches we need two multipliers, offset by 90^o in phase. After multiplication, the resulting signals are integrated from 0 to T and then squared before the squares are added. The difference of the two branches is still fed into a decision device.

Computation of the probability of error is difficult and lengthy; see ECE 630 notes for details. The resulting probability of error is (approximately):

$$P_e = \frac{1}{2}\exp(-\frac{A_0^2 T}{N_0}\frac{4}{\pi^2}).$$

4. Problem 4

Begin by determining the base-band equivalent signals for $x(t)$ and $h(t)$:

$$x_b(t) = \frac{1}{\sqrt{2}}\Pi(\frac{t}{\tau}) \cdot \exp(j2\pi\Delta f t)$$

and

$$h_B(t) = a\exp(-at)u(t).$$

Then, the baseband equivalent output signal $y_b(t)$ is obtained by convolving $x_b(t) * h_b(t)$. Specifically,

$$y_b(t) = \int_0^\infty x_B(t-z) \cdot h_b(z)dz.$$

Evaluation of this integral (is straighforward but tedious and) yields

$$y_b(t) = \frac{1}{\sqrt{2}}\begin{cases} 0 & \text{for } t \leq -\frac{\tau}{2} \\ \exp(j2\pi\Delta f t) - \exp(-at)\cdot\exp(-(a+j2\pi\Delta f)\tau/2) & \text{for } -\frac{\tau}{2} < t < \frac{\tau}{2} \\ \exp(-at)(\exp((a+j2\pi\Delta f)\tau/2) - \exp(-(a+j2\pi\Delta f)\tau/2)) & \text{for } t \geq \frac{\tau}{2} \end{cases}$$

The (passband) output signal $y(t)$ is obtained from the baseband equivalent signal $y_b(t)$ via

$$y(t) = \sqrt{2}\Re\{y_b(t)\cdot\exp(j2\pi f_0 t)\},$$

which reduces to

$$y(t) = \begin{cases} 0 & \text{for } t \leq -\frac{\tau}{2} \\ \cos(2\pi(f_0+\Delta f)t) - \exp(-a(t+\tau/2))\cdot\cos(2\pi(f_0 t - \Delta f\tau/2)) & \text{for } -\frac{\tau}{2} < t < \frac{\tau}{2} \\ \exp(-a(t-\tau/2))\cos(2\pi(f_0 t + \Delta f\tau/2)) - & \\ \exp(-a(t+\tau/2))\cos(2\pi(f_0 t - \Delta f\tau/2)) & \text{for } t \geq \frac{\tau}{2} \end{cases}$$