ECE 630: Statistical Communication Theory
Prof. B.-P. Paris
Homework 8
Due: April 2, 2019

Reading Madhow: Section 3.2 through 3.5.

Problems

1. **Binary Hypothesis Testing**
   The two hypotheses are of the form:

$$
\begin{aligned}
H_0 : \quad p_R(r) &= \frac{1}{2} e^{-|r|} \\
H_1 : \quad p_R(r) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{r^2}{2}}
\end{aligned}
$$

   (a) Find the likelihood ratio.
   (b) Compute the decision regions for various values of the threshold in the likelihood ratio test (i.e., the MPE decision rule).

2. A communications systems gives rise to the following decision problem. Equally likely signals take on the values $s_1 = -s_0 = \sqrt{E}$. There are two independent, Gaussian random variables, N_1 and N_2 each with zero mean and variance σ^2. The following inputs are observed:

$$
\begin{aligned}
R_1 = &\quad s + N_1 \\
R_2 = &\quad N_1 + N_2
\end{aligned}
$$

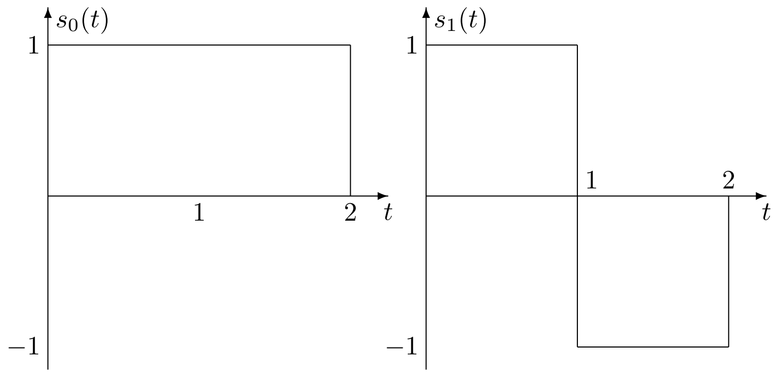
   where s is either s_0 or s_1.

   (a) Show that the MPE decision rule has the form

$$
\begin{aligned}
R_1 + aR2 > \gamma &\quad \text{decide } s_1 \\
R_1 + aR2 < \gamma &\quad \text{decide } s_0
\end{aligned}
$$

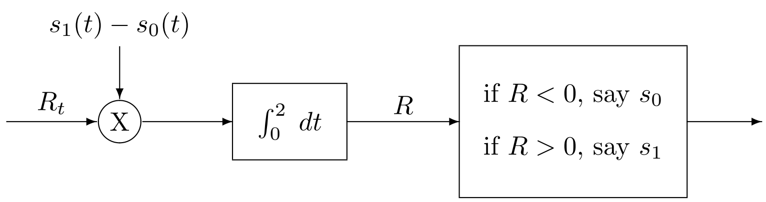
   (b) What is the optimum value of a?
   (c) What is the optimum threshold γ?
   (d) Compute the minimum probability of error.
   (e) By how much would E need to be increased to achieve the same probability of error if only R_1 was observed?

3. The following signals are used to transmit equally likely messages over a channel corrupted by additive, white Gaussian noise of spectral height $\frac{N_0}{2}$:

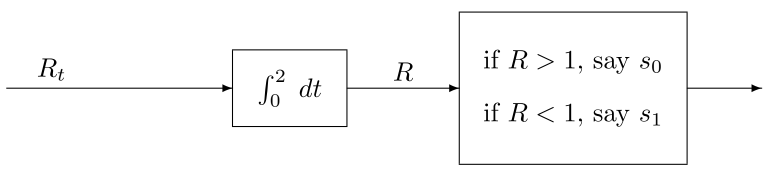


Compute the probability of error attained by the following receivers.

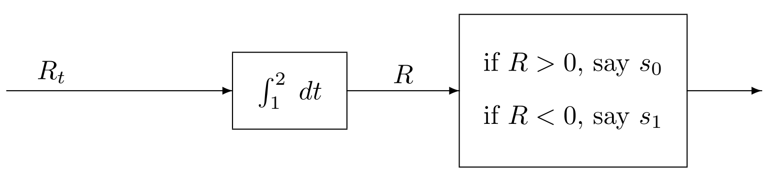
(a)



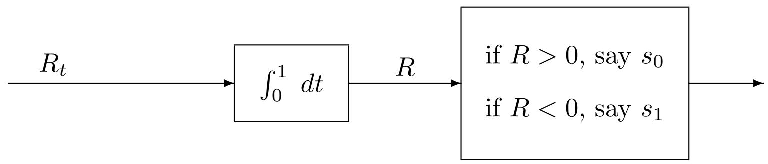
(b)



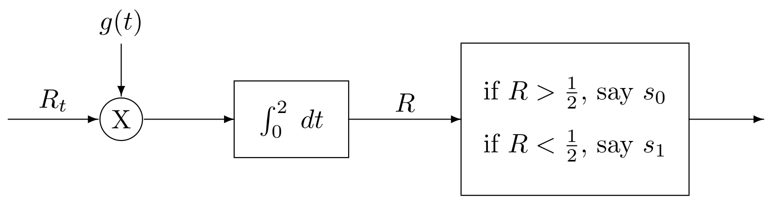
(c)



(d)



(e) Let $g(t) = 1 - |t-1|$, $0 \leq t \leq 2$.



4. In this problem, we analyze the dependence of the probability of error on the threshold of the comparator in the optimum receiver. Assume one of two equally likely messages is transmitted using the following signals,

$$\begin{aligned} s_0(t) &= 0 \qquad \text{for } 0 \leq t \leq T \\ s_1(t) &= \sqrt{\frac{E}{T}} \quad \text{for } 0 \leq t \leq T. \end{aligned}$$

The channel is corrupted by additive, white Gaussian noise of spectral height $\frac{N_0}{2}$.

(a) Draw the block diagram of the optimum receiver.

(b) Compute the probability of error attained by this receiver.

(c) The threshold of the optimum receiver is given by $\gamma = \frac{1}{2}(||s_1||^2 - ||s_0||^2)$. What is the probability of error if instead this threshold were chosen as $\gamma = \lambda||s_1||^2 - (1-\lambda)||s_0||^2$?

(d) Plot the probability of error computed in part (c) for $0 \leq \lambda \leq 1$. Use $\frac{N_0}{2} = 1$, $E = 1$, $T = 1$. You may approximate $Q(x)$ by $\frac{1}{2}\exp(\frac{-x^2}{2})$.