ECE 630: Statistical Communication Theory
Prof. B.-P. Paris
Homework 7
Due: March 26, 2018

Reading Madhow: Section 3.2 and 3.3.

Problems

1. Triangular signals are used to transmit equally likely, binary messages over an AWGN channel with spectral height $\frac{N_0}{2}$. Specifically, the two signals are

$$s_0(t) = \begin{cases} \frac{2A}{T} \cdot t & \text{for } 0 \leq t \leq \frac{T}{2} \\ 2A - \frac{2A}{T} \cdot t & \text{for } \frac{T}{2} \leq t \leq T \\ 0 & \text{else} \end{cases} \qquad s_1(t) = -s_0(t)$$

with amplitude $A = \sqrt{\frac{3E}{T}}$.

We will be comparing the performance of different receiver frontends. Each of the frontends is of the form

$$R = \langle R_t, g(t) \rangle = \int_0^T R_t g(t)\, dt.$$

The backend decides in all cases

$$\hat{m} = \begin{cases} 0 & \text{if } R > 0 \\ 1 & \text{if } R < 0. \end{cases}$$

(a) Compute the energy of signals $s_0(t)$ and $s_1(t)$,

(b) Compute the probability of error if the receiver frontend uses $g(t) = 1$ for $0 \leq t \leq T$.

(c) Compute the probability of error for

$$g(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq T/2 \\ -1 & \text{for } T/2 \leq t \leq T \\ 0 & \text{else.} \end{cases}$$

(d) Compute the probability of error for $g(t) = s_0(t)$.

(e) Explain (in terms of projections) why some of the above receivers are better than others.

2. The following signals are used to communicate one of two equally likely messages over an AWGN channel with spectral height $\frac{N_0}{2}$

$$s_0(t) = \begin{cases} \frac{A}{\sqrt{T}} & \text{for } 0 \leq t \leq T \\ 0 & \text{else} \end{cases} \qquad s_1(t) = \begin{cases} \frac{-1}{\sqrt{T}} & \text{for } 0 \leq t \leq T \\ 0 & \text{else} \end{cases}$$

where $A > 0$.

The receiver frontend computes the integral of the received signal

$$R = \int_0^T R_t \, dt$$

and the backend decides which signal was transmitted using the following decision rule

$$\hat{m} = \begin{cases} 0 & \text{if } R > \gamma \\ 1 & \text{if } R < \gamma \end{cases}$$

where the threshold $\gamma > 0$ is the subject of this problem.

(a) Compute the probability of error when $A = 1$ and $\gamma = 0$.

(b) Compute the probability of error when $A = 3$ and $\gamma = 0$.

(c) When $A = 3$, is there a value of γ that leads to a smaller probability of error? If so, what is the best value of γ and the corresponding probaility of error?

(d) Establish a general relationship between the best value for the threshold γ and the amplitude A. Also, find the corresponding probability of error.

3. The following signals are used to communicate one of two messages over an AWGN channel with spectral height $\frac{N_0}{2}$

$$s_0(t) = \begin{cases} \sqrt{\frac{E_b}{T}} & \text{for } 0 \leq t \leq T \\ 0 & \text{else} \end{cases} \quad s_1(t) = \begin{cases} -\sqrt{\frac{E_b}{T}} & \text{for } 0 \leq t \leq T \\ 0 & \text{else.} \end{cases}$$

The a priori probabilities for signals $s_0(t)$ and $s_1(t)$ are $\pi_0 = \frac{3}{4}$ and $\pi_1 = \frac{1}{4}$, respectively.

The receiver frontend computes the integral of the received signal

$$R = \int_0^T R_t \, dt.$$

(a) Find the average probability of error when the decision rule is

$$\hat{m} = \begin{cases} 0 & \text{if } R > 0 \\ 1 & \text{if } R < 0 \end{cases}$$

(b) Assume now that the decision rule is

$$\hat{m} = \begin{cases} 0 & \text{if } R > \gamma \\ 1 & \text{if } R < \gamma. \end{cases}$$

Give an expression for the average probability of error in terms of the threshold γ.

(c) Minimize the average probability of error with respect to the threshold γ, i.e., find the optimum threshold $\hat{\gamma}$.

(d) Let $p_{R|m=0}(r)$ and $p_{R|m=1}(r)$ denote the conditional pdfs of R. Plot $\pi_0 p_{R|m=0}(r)$ and $\pi_1 p_{R|m=1}(r)$. Describe how the optimum threshold $\hat{\gamma}$ is evident in your plot.

4. A "stealthy" communication system works as follows. To transmit $m = 0$, the transmitter sends white Gaussian noise of spectral height $\frac{E}{2}$ for T seconds. The transmitter does not transmit a signal to send $m = 1$ (i.e., $s_1(t) = 0$). Both messages are equally likely. The channel adds white Gaussian noise with spectral height $\frac{N_0}{2}$.

   Assume that the (not optimal) receiver frontend computes the integral of the received signal

$$R = \int_0^T R_t \, dt.$$

   (a) Find the conditional densities of R for both $m = 0$ and $m = 1$.

   (b) What is the error probability when the decision rule is

$$\hat{m} = \begin{cases} 0 & \text{if } R > 0 \\ 1 & \text{if } R < 0 \end{cases}$$

   (c) Assume now that $E = 5N_0$ and that $T = 1$. Compute the probability of error for the decision rule

$$\hat{m} = \begin{cases} 0 & \text{if } R^2 > \gamma \\ 1 & \text{if } R^2 < \gamma, \end{cases}$$

   with $\gamma = N_0 T \cdot \ln(\frac{E}{N_0})$. Note the square in the decision rule!