ECE 630: Statistical Communication Theory
Prof. B.-P. Paris
Homework 5
Due: March 5, 2019

Reading Madhow: Section 3.2.

Problems

1. Let x and y be elements of a normed linear vector space.
   (a) Determine whether the following are valid inner products for the indicated space.
      i. $\langle x, y\rangle = \underline{x}^T A \underline{y}$, where A is a nonsingular,NxN matrix and $\underline{x}$, $\underline{y}$ are elements of the space of N-dimensional vectors.
      ii. $\langle x, y\rangle = \underline{x}\underline{y}^T$, where $\underline{x}$ and $\underline{y}$ are elements of the space of N-dimensional (column!) vectors.
      iii. $\langle x, y\rangle = \int_0^T x(t)y(T-t)\,dt$, where x and y are finite energy signals defined over $[0, T]$.
      iv. $\langle x, y\rangle = \int_0^T w(t)x(t)y(t)\,dt$, where x and y are finite energy signals defined over $[0, T]$ and $w(t)$ is a non-negative function.
      v. $E[XY]$, where X and Y are real-valued random variables having finite mean-square values.
      vi. $\text{Cov}(X, Y)$, the covariance of the real-valued random variables X and Y. Assume that X and Y have finite mean-square values.
   (b) Under what conditions is

$$\int_0^T \int_0^T Q(t, u)x(t)y(u)\,dt\,du$$

   a valid inner product for the space of finite-energy functions defined over $[0, T]$?

2. Let $x(t)$ be a signal of finite energy over the interval $[0, T]$. In other words, $x(t)$ is a vector in the Hilbert space $L_2(0, T)$. Signals may be complex values, so that the appropriate inner product is

$$\langle x, y\rangle = \int_0^T x(t) \cdot y^*(t)\,dt.$$

   Consider subspace $\mathcal{L}$ of $L_2(0, T)$ that consists of signals of the form

$$y_n(t) = X_n \exp(j2\pi nt/T) \text{ for } 0 \leq t \leq T,$$

   where X_n may be complex valued.
   (a) Find the signal $\hat{y}_n(t)$ that best approximates the signal $x(t)$, i.e., $\hat{y}_n(t)$ minimizes $\|x - y_n\|$ among all elements of $\mathcal{L}$.
   Hint: Find the best complex amplitude $\hat{X}_n$.

(b) Now define the *error* signal $z(t) = x(t) - \hat{y}_n(t)$. Show that $z(t)$ is orthogonal to the subspace $\mathcal{L}$, i.e., it is orthogonal to all elements of $\mathcal{L}$.

(c) How do the above results illustrate the projection theorem?

3. **Linear Regression**
The elements of a vector of random variables $\vec{Y}$ follow the model

$$Y_n = ax_n + b + N_n$$

where x_n are known and N_n are zero mean, iid Gaussian noise samples with variance σ^2. The parameters a and b are to be determined. We can think of the solution to this problem as the projection of $\vec{Y}$ onto the subspace spanned by $a\vec{x} + b$

(a) Determine the *least-squares estimates* for a and b, i.e., find

$$\hat{a}, \hat{b} = \arg\min_{a,b} \|\vec{Y} - (a\vec{x} + b)\|^2.$$

(b) What are the expected values of these estimates, $\mathrm{E}[\hat{a}]$ and $\mathrm{E}[\hat{b}]$?

(c) Compute $\hat{a}$ and $\hat{b}$, when data are given by the (x_n, Y_n) pairs

$$\{(x_n, Y_n)\}_{n=1}^{5} = \{(0, 1.3), (1, 0.2), (2, 0.1), (3, -0.4), (4, -1.2)\}.$$

(d) Is it true that the least-squares estimates for a and b are given by the inner products

$$\hat{a} = \langle \vec{Y}, \vec{x} \rangle \text{ and } \hat{b} = \langle \vec{Y}, \vec{1} \rangle?$$

$\vec{1}$ denotes a vector of 1's. Explain why or why not?