ECE 630: Statistical Communication Theory
Prof. B.-P. Paris
Homework 4
Due: February 26, 2019

Reading Madhow: Section 3.3.

Problems

1. The stationary random process X_t is passed through a linear filter with transfer function $H(f)$,

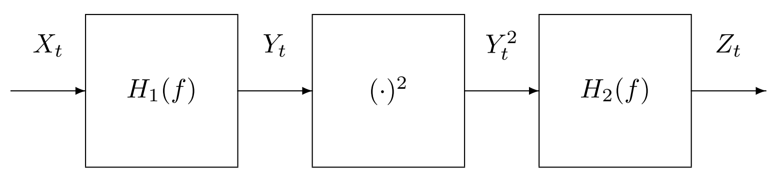
$$H(f) = \frac{j2\pi f + a}{j2\pi f + 2a}.$$

The output process is labeled Y_t. The mean of Y_t is measured to be $\frac{1}{2}$ and the covariance function of Y_t is found to be

$$K_Y(\tau) = a^2 e^{-2a|\tau|}.$$

   (a) Compute the power spectral density of Y_t.
   (b) Find the second order description of X_t.

2. In practice one often wants to measure the power spectral density of a stochastic process. For the purposes of this problem, assume the process X_t is wide-sense stationary, zero mean, and Gaussian. The following measurement system is proposed.



Here $H_1(f)$ is the transfer function of an ideal bandpass filter and $H_2(f)$ is an ideal lowpass,

$$H_1(f) = \begin{cases} 1 & \text{for } f_0 - \frac{\Delta f}{2} \leq |f| \leq f_0 + \frac{\Delta f}{2} \\ 0 & \text{else} \end{cases}$$

$$H_2(f) = \begin{cases} \frac{1}{2\Delta f} & \text{for } |f| \leq \Delta f \\ 0 & \text{else.} \end{cases}$$

Assume that Δf is small compared to the range of frequencies over which $S_X(f)$ varies, i.e., you may assume that $S_X(f)$ is constant over intervals of width Δf.

   (a) Find the mean and correlation function of Y_t^2 in terms of the second order description of X_t. The following may be helpful — this is known as Isserlin's Theorem: If X and Y are jointly Gaussian, then $\mathrm{E}[X^2Y^2] = \mathrm{E}[X^2]\mathrm{E}[Y^2] + 2\mathrm{E}^2[XY]$

(b) Compute the the power spectral density of the process Z_t.

(c) Compute the expected value of Z_t.

(d) By considering the variance of Z_t, comment on the accuracy of this measurement of the power density of the process X_t.

3. Let W_t (for $t \geq 0$) be a Wiener process (Brownian motion) with variance σ^2. Define the random process X_t as the (runnning) integral over W_t, i.e., for $t \geq 0$

$$X_t = \int_0^t W_s \, ds.$$

(a) Find the mean of X_t.

(b) Compute the autocorrelation function of X_t.

(c) Is W_t wide-sense stationary?

(d) Compute the following probability for $t \geq 0$

$$\Pr\{|X_t| > \sigma \cdot t\}.$$