ECE 460: Communication & Information Theory
Midterm Exam Solution – Problem 1
Willy W Huaracha
April 1, 2002

Sampling:

Throughout this problem consider the following signal

$$s(t) = f_1 .\sin c(2\pi f_1 t).f_2 .\sin c(2\pi f_2 t)$$

where $\sin c(x) = \sin(x)/x$. Assume that $f2>f1$. Also, let $\Pi(x)$ denote the rectangular pulse defined, i.e.,

$$\Pi(x) = \begin{cases} 1, & for\ |x| < \frac{1}{2} \\ 0, & elsewhere \end{cases}$$

(a) Show that the inverse Fourier transform of $\Pi(f/fo)$ is given by $f_o \sin c(\pi f_o t)$.

$$X(f) = \Pi(f/f_o)$$

$$x(t) = \int_{-fo/2}^{fo/2} e^{j2\pi ft} df$$

$$x(t) = \frac{1}{j2\pi t}\left[e^{j2\pi ft}\right]_{-fo/2}^{fo/2} = \frac{1}{j2\pi t}\left[e^{j\pi fot} - e^{-j\pi fot}\right]$$

$$x(t) = \frac{1}{\pi t}\sin(\pi f_o t) = f_o \sin c(\pi f_o t)$$

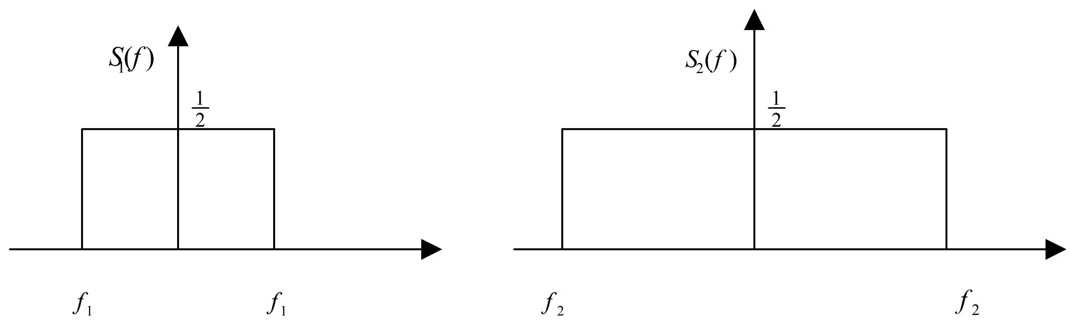
(b) Use the convolution rule, to find the Fourier transform $S(f)$ of $s(t)$. Plot the magnitude of $S(f)$. Label your plot very accurately! What is the value of $S(f_2)$?

$$s(t) = f_1 .\sin c(2\pi f_1 t).f_2 .\sin c(2\pi f_2 t)$$

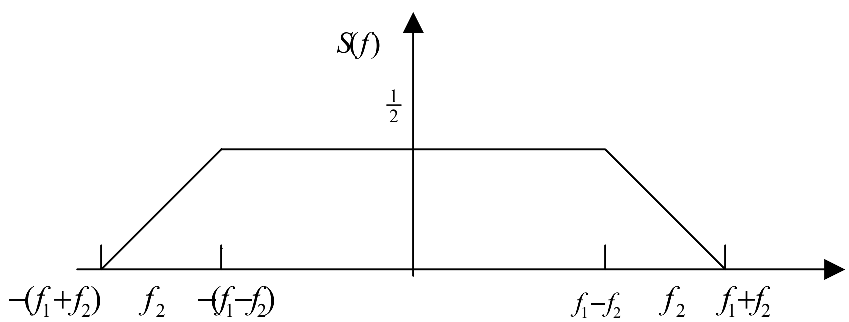
First we take the Fourier transform of $s(t)$ as and obtain the convolution of two rectangular pulses.

$$S(f) = \frac{1}{2}\Pi\left(\frac{f}{2f_1}\right) * \frac{1}{2}\Pi\left(\frac{f}{2f_2}\right)$$

The two rectangular pulses are plotted below:



By convoluting the above rectangular pulses we can determine the plot of the $S(f)$ as shown below.



$$S(f)=\begin{cases}\frac{f}{2f_1}+(f_2+f_1), & for \quad -(f_2+f_1)<f<-(f_2-f_1)\\ \frac{1}{2}, & for \quad -(f_2-f_1)<f<(f_2-f_1)\\ \frac{-f}{2f_1}+(f_2+f_1), & for \quad (f_2-f_1)<f<(f_2+f_1)\\ 0, & elsewhere\end{cases}$$

The value of $S(f_2)$ is: $S(f_2)=\frac{1}{2}\prod\left(\frac{f}{2f_2}\right)$

(c) Is it possible to reconstruct completely the signal $s(t)$ from samples taken at rate $1/T_s$? Justify answer! If answer is "yes", give the largest sampling period T_s, which allows for perfect reconstruction.

From the results obtained above for $S(f)$ we can determine the Bandwidth frequency to be $B = (f_2 + f_1)$. Then it is possible to reconstruct the signal completely for the rate of

$$\frac{1}{T_s} \geq 2B = 2(f_2 + f_1) \qquad \text{or} \qquad T_s \leq \frac{1}{2B}$$

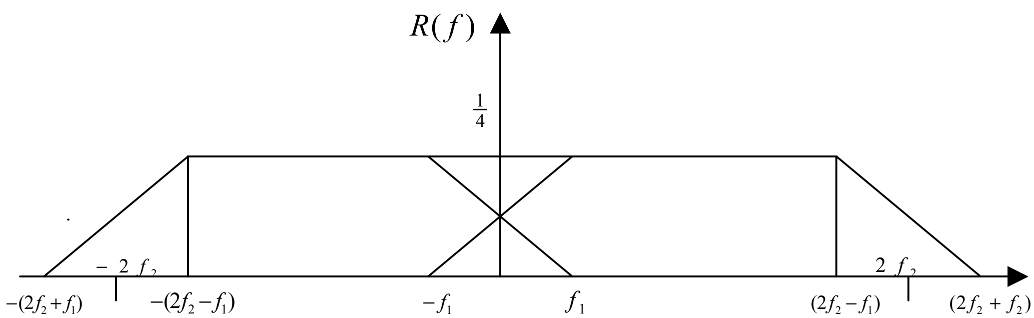
(d) Now, consider the signal $r(t) = s(t).\cos(2\pi f_2 t)$. Compute the Fourier Transform $R(f)$ of $r(t)$. Plot the magnitude of $R(f)$ accurately.

$$r(t) = s(t) \cdot \cos(2\pi f_2 t)$$

$$R(f) = S(f) \cdot \frac{1}{2}\left[\delta(f - f_2) + \delta(f + f_2)\right]$$

$$R(f) = \frac{1}{2}\left[S(f - f_2) + S(f + f_2)\right]$$

Then we can plot the magnitude of $R(f)$ by using the information above.



Then from this plot we can derive the value of $R(f)$ as the convolution of two rectangular pulses.

$$R(f) = \frac{1}{8}\prod\left(\frac{f}{2f_1}\right) * \prod\left(\frac{f}{4f_2}\right)$$

Which can be interpreted as follows:

$$R(f) = \begin{cases} \frac{f}{2f_1} + (2f_2 + f_1), & for \quad -(2f_2 + f_1) < f < -(2f_2 - f_1) \\ \frac{1}{4}, & for \quad -(2f_2 - f_1) < f < (2f_2 - f_1) \\ \frac{-f}{2f_1} + (2f_2 + f_1), & for \quad (2f_2 - f_1) < f < (2f_2 + f_1) \\ 0, & elsewhere \end{cases}$$

(e) Conclude that $r(t)$ can be written in the form $A \sin c(2\pi f_a t) \cdot \sin c(2\pi f_b t)$ with suitably selected constants A, f_a, f_b. Provide values for all three constants and explain how you obtain them.

The following results are obtained by taking the Inverse Fourier Transform of $R(f)$.

$$r(t) = \frac{1}{8}\left[2 f_1 . \sin c\left(2\pi f_1 t\right) . 4 f_2 . \sin c\left(4\pi f_2 t\right)\right]$$

$$r(t) = f_1 . \sin c\left(2\pi f_1 t\right) . f_2 . \sin c\left(4\pi f_2 t\right)$$