Sampling:

Throughout this problem consider the following signal

\[ s(t) = f_1 \cdot \sin(2\pi f_1 t) + f_2 \cdot \sin(2\pi f_2 t) \]

where \( \sin(c(x)) = \sin(x)/x \). Assume that \( f_2 > f_1 \). Also, let \( \Pi(x) \) denote the rectangular pulse defined, i.e.,

\[ \Pi(x) = \begin{cases} 1, & \text{for } |x| < \frac{1}{2} \\ 0, & \text{elsewhere} \end{cases} \]

(a) Show that the inverse Fourier transform of \( \Pi(f/f_o) \) is given by \( f_o \cdot \sin(\pi f_o t) \).

\[ X(f) = \Pi(f/f_o) \]

\[ x(t) = \int_{-f_o/2}^{f_o/2} e^{j2\pi ft} df \]

\[ x(t) = \frac{1}{j2\pi t} \left[ e^{j2\pi f_1 t} \right]_{-f_o/2}^{f_o/2} = \frac{1}{j2\pi t} \left[ e^{j\pi f_1 t} - e^{-j\pi f_1 t} \right] \]

\[ x(t) = \frac{1}{\pi t} \sin(\pi f_o t) = f_o \cdot \sin(\pi f_o t) \]

(b) Use the convolution rule, to find the Fourier transform \( S(f) \) of \( s(t) \). Plot the magnitude of \( S(f) \). Label your plot very accurately! What is the value of \( S(f_2) \)?

\[ s(t) = f_1 \cdot \sin(2\pi f_1 t) + f_2 \cdot \sin(2\pi f_2 t) \]

First we take the Fourier transform of \( s(t) \) as and obtain the convolution of two rectangular pulses.

\[ S(f) = \frac{1}{2} \prod \left( \frac{f}{2f_1} \right) \ast \frac{1}{2} \prod \left( \frac{f}{2f_2} \right) \]

The two rectangular pulses are plotted below:
By convoluting the above rectangular pulses we can determine the plot of the $S(f)$ as shown below.

$$S(f) = \begin{cases} 
\frac{f}{2f_1} + (f_2 + f_1), & \text{for } -(f_2 + f_1) < f < -(f_2 - f_1) \\
\frac{1}{2}, & \text{for } -(f_2 - f_1) < f < (f_2 - f_1) \\
\frac{-f}{2f_1} + (f_2 + f_1), & \text{for } (f_2 - f_1) < f < (f_2 + f_1) \\
0, & \text{elsewhere} 
\end{cases}$$

The value of $S(f_2)$ is: $S(f_2) = \frac{1}{2} \prod_{f_2} \left( \frac{f}{2f_2} \right)$

(c) Is it possible to reconstruct completely the signal $s(t)$ from samples taken at rate $1/T_s$? Justify answer! If answer is “yes”, give the largest sampling period $T_s$, which allows for perfect reconstruction.
From the results obtained above for \( S(f) \) we can determine the Bandwidth frequency to be \( B = (f_2 + f_1) \). Then it is possible to reconstruct the signal completely for the rate of

\[
\frac{1}{T_s} \geq 2B = 2(f_2 + f_1) \quad \text{or} \quad T_s \leq \frac{1}{2B}
\]

(d) Now, consider the signal \( r(t) = s(t) \cdot \cos(2\pi f_z t) \). Compute the Fourier Transform \( R(f) \) of \( r(t) \). Plot the magnitude of \( R(f) \) accurately.

\[
r(t) = s(t) \cdot \cos(2\pi f_z t)
\]

\[
R(f) = S(f) \cdot \frac{1}{2} \left[ \delta(f - f_z) + \delta(f + f_z) \right]
\]

\[
R(f) = \frac{1}{2} \left[ S(f - f_z) + S(f + f_z) \right]
\]

Then we can plot the magnitude of \( R(f) \) by using the information above.

Then from this plot we can derive the value of \( R(f) \) as the convolution of two rectangular pulses.

\[
R(f) = \frac{1}{8} \prod \left( \frac{f}{2f_1} \right) \ast \prod \left( \frac{f}{4f_2} \right)
\]

Which can be interpreted as follows:

\[
R(f) = \begin{cases} 
\frac{f}{2f_1} + (2f_2 + f_1), & \text{for} \quad -2(f_2 + f_1) < f < -(2f_2 - f_1) \\
\frac{1}{2f_1}, & \text{for} \quad -(2f_2 - f_1) < f < (2f_2 - f_1) \\
\frac{f}{2f_1} + (2f_2 + f_1), & \text{for} \quad (2f_2 - f_1) < f < (2f_2 + f_1) \\
0, & \text{elsewhere}
\end{cases}
\]
(e) Conclude that \( r(t) \) can be written in the form \( A \sin c(2\pi f_a t) \cdot \sin c(2\pi f_b t) \) with suitably selected constants \( A, f_a, f_b \). Provide values for all three constants and explain how you obtain them.

The following results are obtained by taking the Inverse Fourier Transform of \( R(f) \).

\[
r(t) = \frac{1}{8} \left[ 2 f_1 \sin c(2\pi f_1 t), 4 f_2 \sin c(4\pi f_2 t) \right]
\]

\[
r(t) = f_1 \sin c(2\pi f_1 t), f_2 \sin c(4\pi f_2 t)
\]