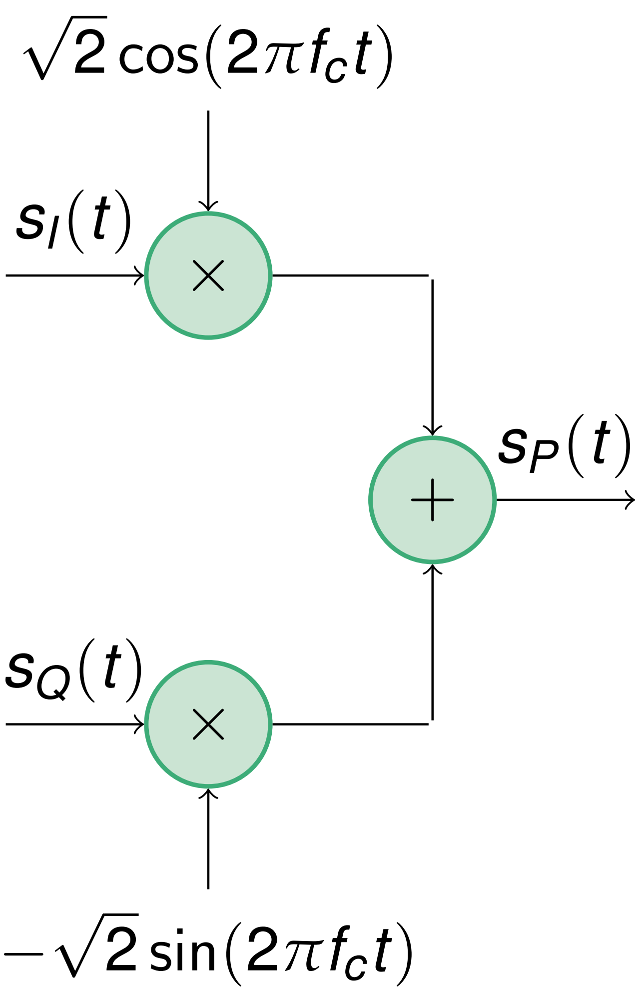
Passband Signals

- We have seen that many signal sets include both $\sin(2\pi f_c t)$ and $\cos(2\pi f_c t)$.
  - Examples include PSK and QAM signal sets.
- Such signals are referred to as passband signals.
  - Passband signals have frequency spectra concentrated around a carrier frequency f_c.
  - This is in contrast to baseband signals with spectrum centered at zero frequency.
- Baseband signals can be converted to passband signals through up-conversion.
- Passband signals can be converted to baseband signals through down-conversion.

Up-Conversion



- The passband signal $s_P(t)$ is constructed from two (digitally modulated) baseband signals, $s_I(t)$ and $s_Q(t)$.
  - Note that two signals can be carried simultaneously!
    - $s_I(t)$ and $s_Q(t)$ are the in-phase (I) and quadrature (Q) compenents of $s_p(t)$.
  - This is a consequence of $s_I(t)\cos(2\pi f_c t)$ and $s_Q(t)\sin(2\pi f_c t)$ being orthogonal
    - when the carrier frequency f_c is much greater than the bandwidth of $s_I(t)$ and $s_Q(t)$.

Exercise: Orthogonality of In-phase and Quadrature Signals

- Show that $s_I(t)\cos(2\pi f_c t)$ and $s_Q(t)\sin(2\pi f_c t)$ are orthogonal when $f_c \gg B$, where B is the bandwidth of $s_I(t)$ and $s_Q(t)$.
  - You can make your argument either in the time-domain or the frequency domain.

Baseband Equivalent Signals

- The passband signal $s_P(t)$ can be written as

$$s_P(t) = \sqrt{2} s_I(t) \cdot \cos(2\pi f_c t) - \sqrt{2} s_Q(t) \cdot \sin(2\pi f_c t).$$

- If we define $s(t) = s_I(t) + j \cdot s_Q(t)$, then $s_P(t)$ can also be expressed as

$$\begin{aligned} s_P(t) &= \sqrt{2} \cdot \Re\{s(t)\} \cdot \cos(2\pi f_c t) - \sqrt{2} \cdot \Im\{s(t)\} \cdot \sin(2\pi f_c t) \\ &= \sqrt{2} \cdot \Re\{s(t) \cdot \exp(j2\pi f_c t)\}. \end{aligned}$$

- The signal $s(t)$:
  - is called the baseband equivalent, or the complex envelope of the passband signal $s_P(t)$.
  - It contains the same information as $s_P(t)$.
  - Note that $s(t)$ is *complex-valued*.

Polar Representation

- Sometimes it is useful to express the complex envelope $s(t)$ in polar coordinates:

$$\begin{aligned} s(t) &= s_I(t) + j \cdot s_Q(t) \\ &= e(t) \cdot \exp(j\theta(t)) \end{aligned}$$

with

$$\begin{aligned} e(t) &= \sqrt{s_I^2(t) + s_Q^2(t)} \\ \tan\theta(t) &= \frac{s_Q(t)}{s_I(t)} \end{aligned}$$

- Also,

$$\begin{aligned} s_I(t) &= e(t) \cdot \cos(\theta(t)) \\ s_Q(t) &= e(t) \cdot \sin(\theta(t)) \end{aligned}$$

Exercise: Complex Envelope

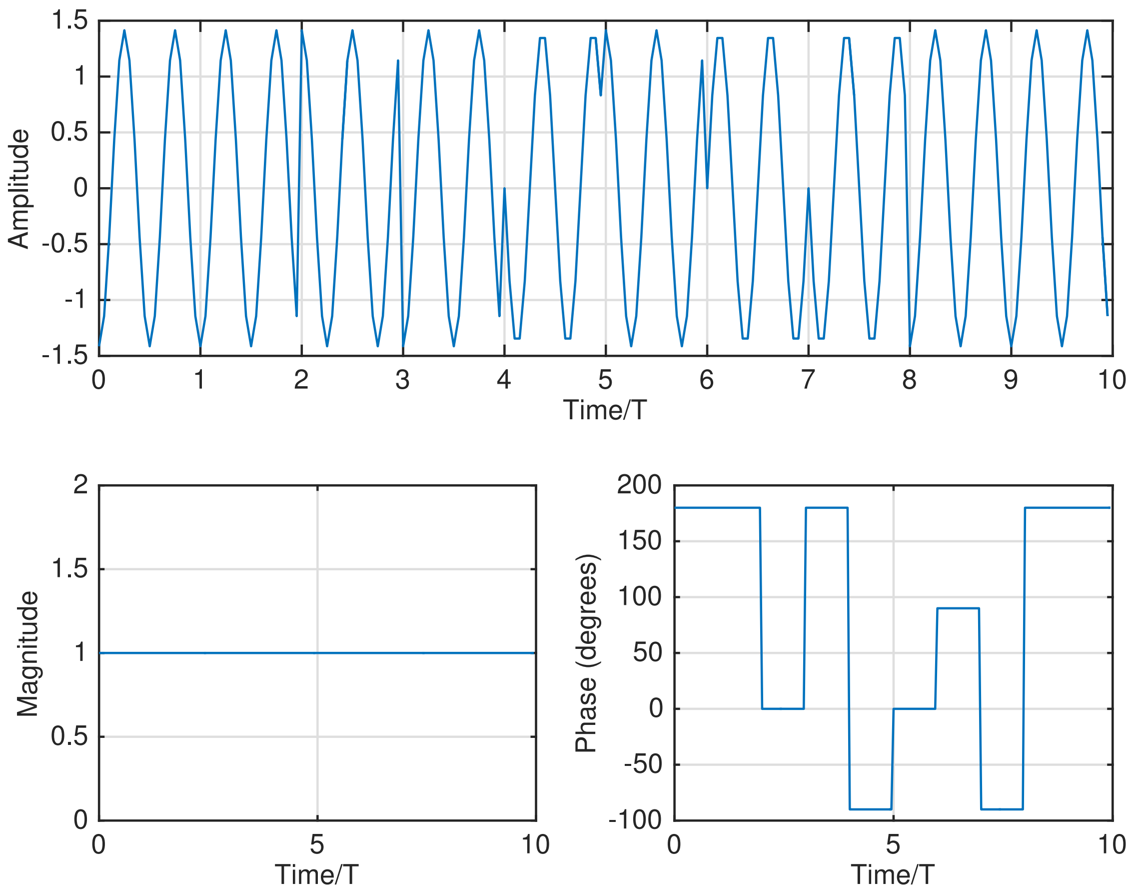
- Find the complex envelope representation of the signal

$$s_p(t) = \text{sinc}(t/T)\cos(2\pi f_c t + \frac{\pi}{4}).$$

- **Answer:**

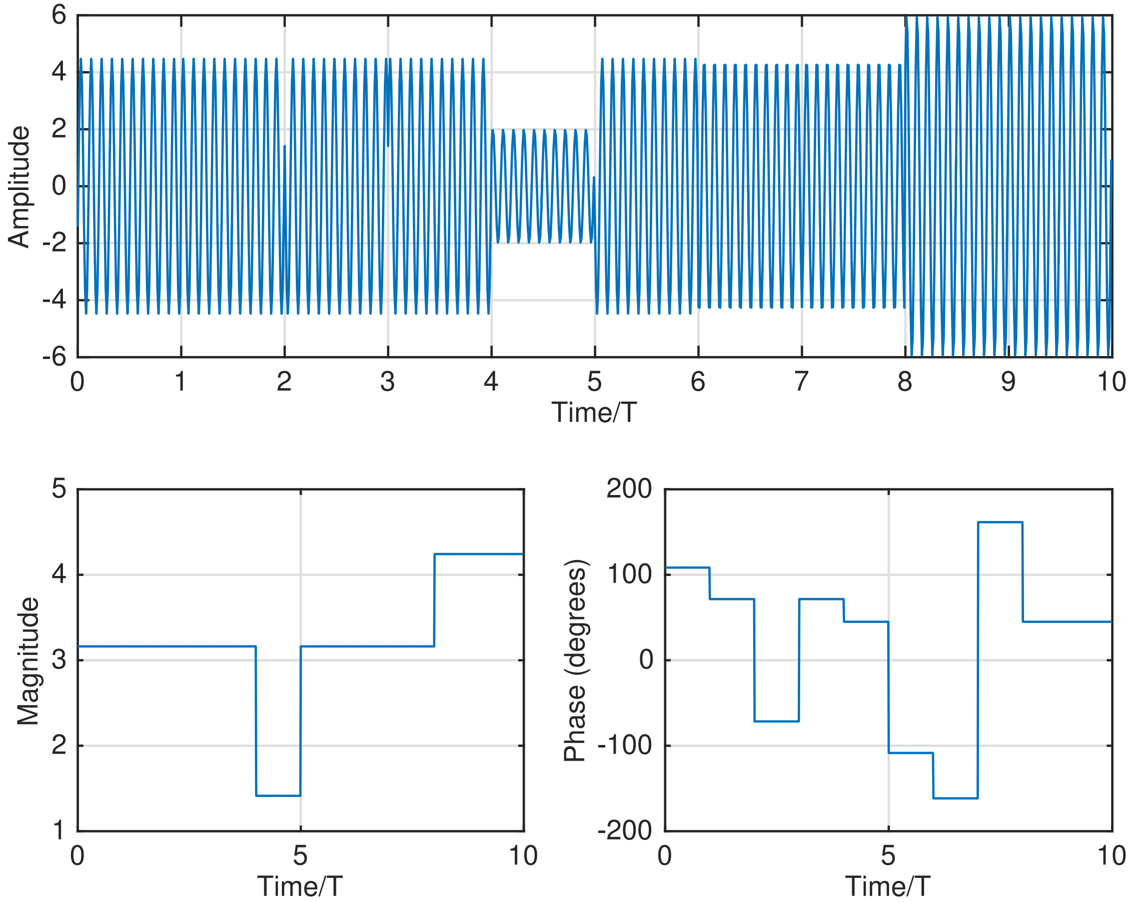
$$\begin{aligned} s(t) &= \frac{e^{j\pi/4}}{\sqrt{2}}\text{sinc}(t/T) \\ &= \frac{1}{2}(\text{sinc}(t/T) + j\text{sinc}(t/T)). \end{aligned}$$

Illustration: QPSK with $f_c = 2/T$



- Passband signal (top): segments of sinusoids with different phases.
  - Phase changes occur at multiples of T.
- Baseband equivalent signal (bottom) is complex valued; magnitude and phase are plotted.
  - Magnitude is constant (rectangular pulses).
- Complex baseband signal shows symbols much more clearly than passband signal.

Illustration: 16-QAM with $f_c = 10/T$



- Passband signal (top): segments of sinusoids with different phases.
  - Phase and amplitude changes occur at multiples of T.
- Baseband signal (bottom) is complex valued; magnitude and phase are plotted.

Frequency Domain

- The time-domain relationships between the passband signal $s_p(t)$ and the complex envelope $s(t)$ lead to corresponding frequency-domain expressions.
- Note that

$$
\begin{aligned}
s_p(t) &= \Re\{s(t) \cdot \sqrt{2}\exp(j2\pi f_c t)\} \\
&= \frac{\sqrt{2}}{2}\left(s(t) \cdot \exp(j2\pi f_c t) + s^*(t) \cdot \exp(-j2\pi f_c t)\right).
\end{aligned}
$$

- Taking the Fourier transform of this expression:

$$
S_P(f) = \frac{\sqrt{2}}{2}\left(S(f - f_c) + S^*(-f - f_c)\right).
$$

  - Note that $S_P(f)$ has the conjugate symmetry ($S_P(f) = S_P^*(-f)$) that real-valued signals must have.

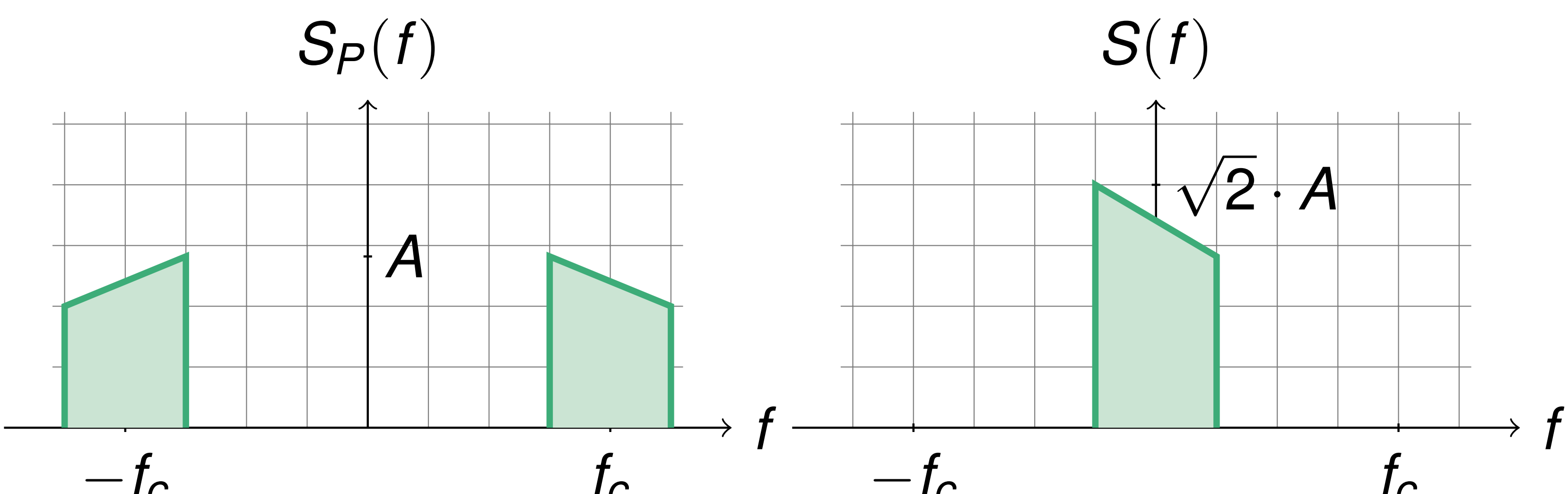
Frequency Domain

- In the frequency domain:

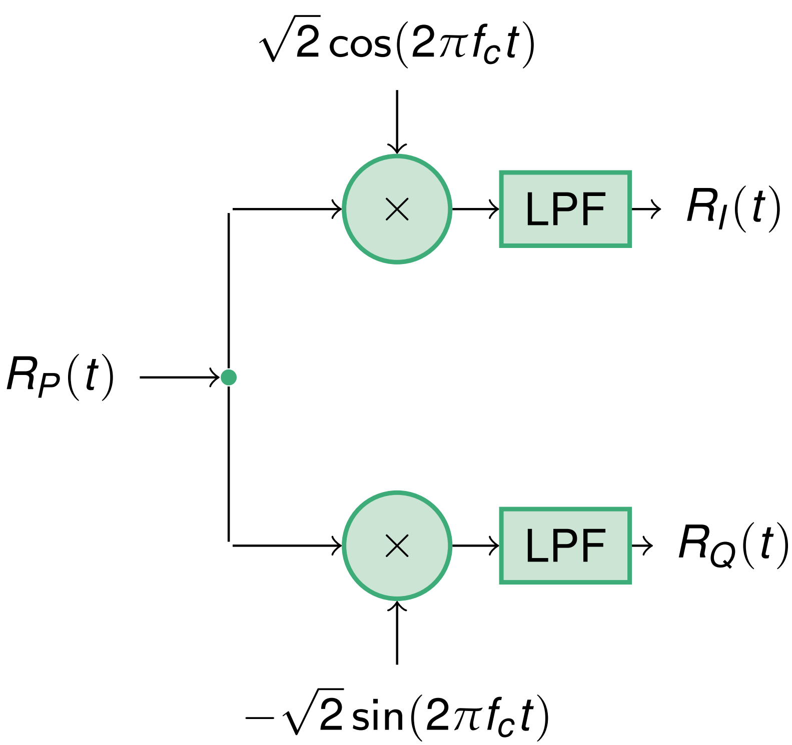
$$S_P(f) = \frac{\sqrt{2}}{2}\left(S(f-f_c) + S^*(-f-f_c)\right).$$

and, thus,

$$S(f) = \begin{cases} \sqrt{2}\cdot S_P(f+f_c) & \text{for } f+f_c>0 \\ 0 & \text{else.} \end{cases}$$



Down-conversion



- The down-conversion system is the mirror image of the up-conversion system.
- The top-branch recovers the *in-phase* signal $s_I(t)$.
- The bottom branch recovers the *quadrature* signal $s_Q(t)$
  - See next slide for details.

Down-Conversion

- Let the the passband signal $s_p(t)$ be input to down-coverter:

$$s_P(t) = \sqrt{2}(s_I(t)\cos(2\pi f_c t) - s_Q(t)\sin(2\pi f_c t))$$

- Multiplying $s_P(t)$ by $\sqrt{2}\cos(2\pi f_c t)$ on the top branch yields

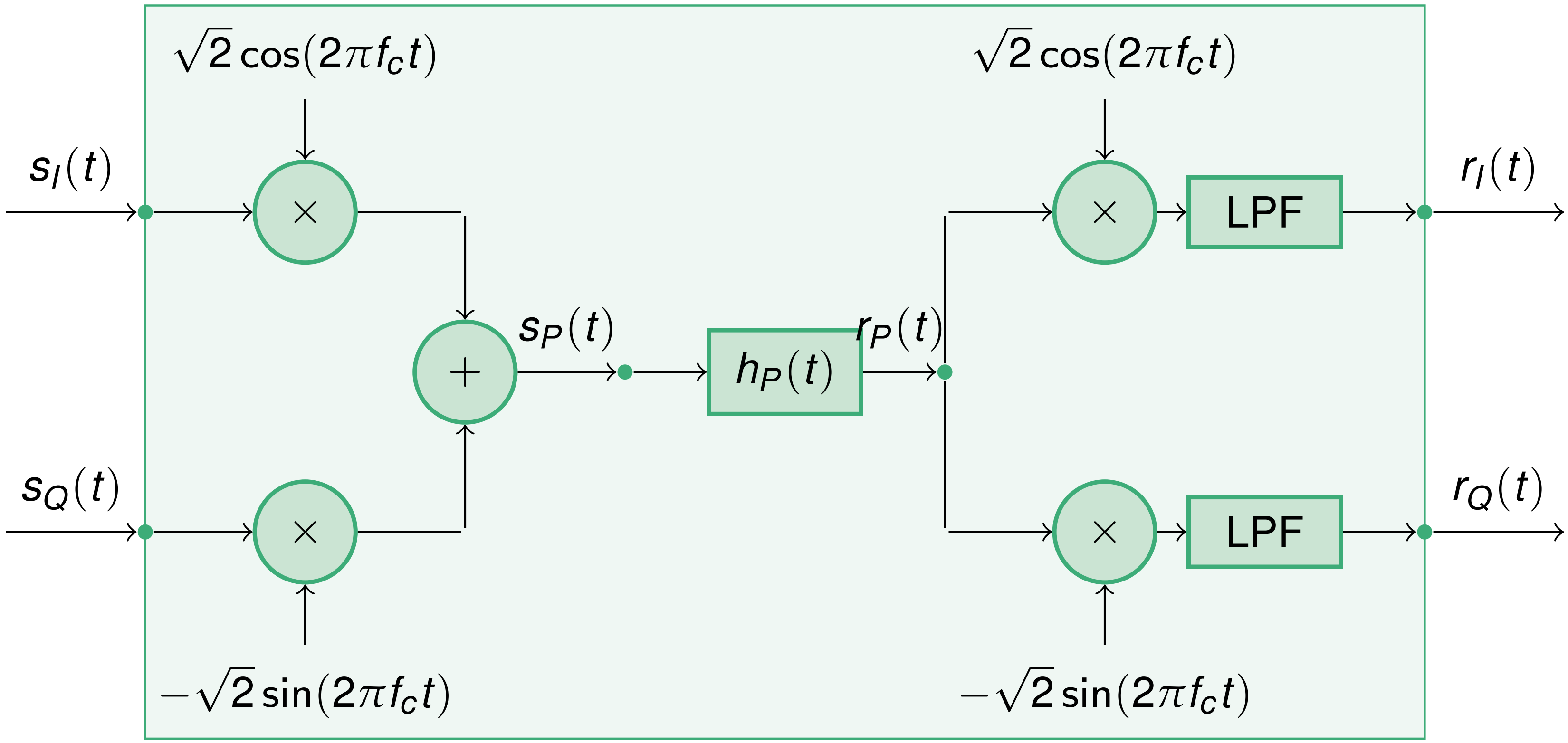
$$\begin{aligned} s_P(t) \cdot \sqrt{2}\cos(2\pi f_c t) & \\ &= 2s_I(t)\cos^2(2\pi f_c t) - 2s_Q(t)\sin(2\pi f_c t)\cos(2\pi f_c t) \\ &= s_I(t) + s_I(t)\cos(4\pi f_c t) - s_Q(t)\sin(4\pi f_c t). \end{aligned}$$

- The low-pass filter rejects the components at $\pm 2f_c$ and retains $s_I(t)$.
- A similar argument shows that the bottom branch yields $s_Q(t)$.

Extending the Complex Envelope Perspective

- The baseband description of the transmitted signal is very convenient:
  - it is more compact than the passband signal as it does not include the carrier component,
  - while retaining all relevant information.
- However, we are also concerned what happens to the signal as it propagates to the receiver.
  - **Question:** Do baseband techniques extend to other parts of a passband communications system?
    - Filtering of the passband signal
    - Noise added to the passband signal

Complete Passband System



- Question: Can the pass band filtering ($h_P(t)$) be described in baseband terms?

Passband Filtering

- For the passband signals $s_P(t)$ and $R_P(t)$

$$r_P(t) = s_P(t) * h_P(t) \quad \text{(convolution)}$$

- Define a baseband equivalent impulse (complex) response $h(t)$.
- The relationship between the passband and baseband equivalent impulse response is

$$h_P(t) = \Re\{h(t) \cdot \sqrt{2} \exp(j2\pi f_c t)\}$$

- Then, the baseband equivalent signals $s(t)$ and $r(t) = r_I(t) + jr_Q(t)$ are related through

$$r(t) = \frac{s(t) * h(t)}{\sqrt{2}} \leftrightarrow R(f) = \frac{S(f)H(f)}{\sqrt{2}}.$$

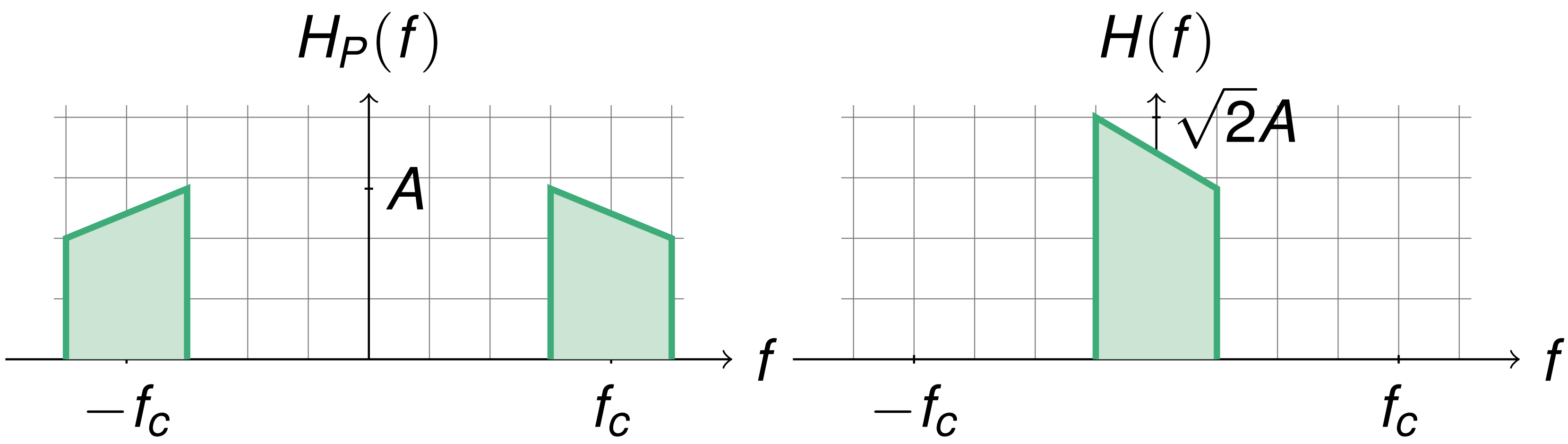
  - Note the division by $\sqrt{2}$!

Passband and Baseband Frequency Response

- In the frequency domain

$$H(f) = \begin{cases} \sqrt{2} H_P(f + f_c) & \text{for } f + f_c > 0 \\ 0 & \text{else.} \end{cases}$$

$$H_p(f) = \frac{\sqrt{2}}{2} \left(H(f - f_c) + H^*(-f - f_c) \right)$$



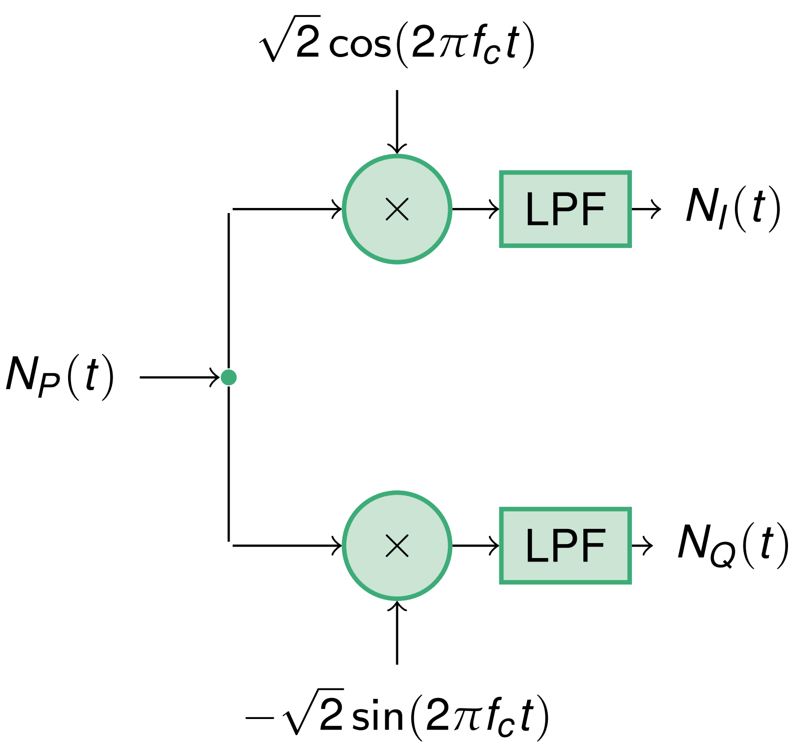
Exercise: Multipath Channel

- A multi-path channel has (pass-band) impulse response

$$h_P(t) = \sum_k a_k \cdot \delta(t - \tau_k).$$

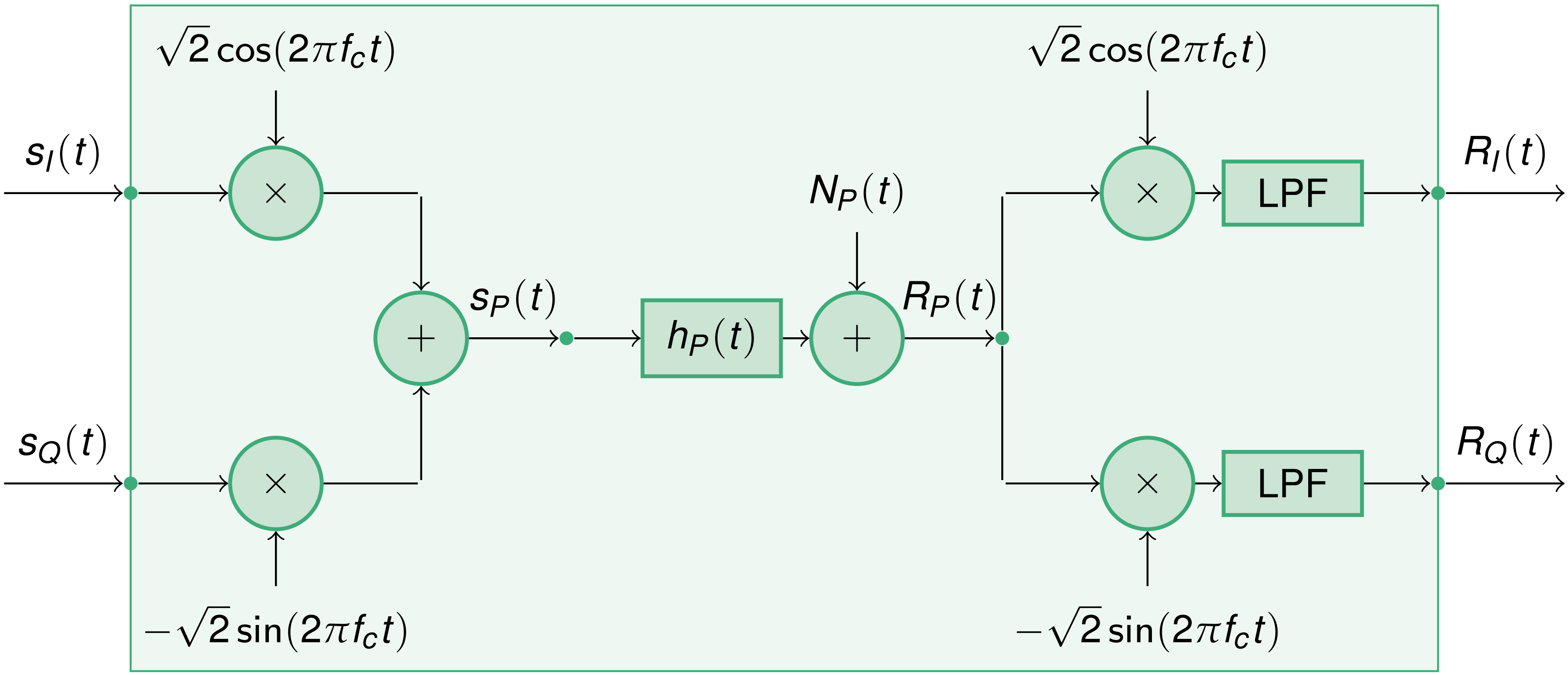
Find the baseband equivalent impulse response $h(t)$ (assuming carrier frequency f_c) and the response to the input signal $s_p(t) = \cos(2\pi f_c t)$.

Passband White Noise



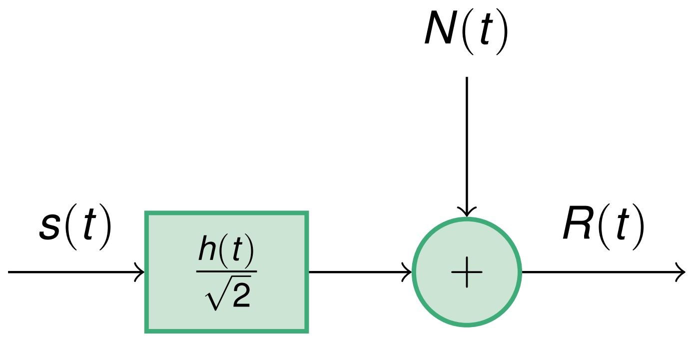
- Let (real-valued) white Gaussian noise $N_P(t)$ of spectral height $\frac{N_0}{2}$ be input to the down-converter.
- Then, each of the two branches produces indepent, white noise processes $N_I(t)$ and $N_Q(t)$ with spectral height $\frac{N_0}{2}$.
- This can be interpreted as (circular) complex noise of spectral height N_0.

Complete Passband System



- Complete pass-band system with channel (filter) and passband noise.

Baseband Equivalent System



- The passband system can be interpreted as follows to yield an equivalent system that employs only baseband signals:
  - baseband equivalent transmitted signal: $s(t) = s_I(t) + j \cdot s_Q(t)$.
  - baseband equivalent channel with complex valued impulse response: $h(t)$.
  - baseband equivalent received signal: $R(t) = R_I(t) + j \cdot R_Q(t)$.
  - complex valued, additive Gaussian noise: $N(t)$ with spectral height N_0.

Generalizing The Optimum Receiver

- We have derived all relationships for the optimum receiver for real-valued signals.
- When we use complex envelope techniques, some of our expressions must be adjusted.
  - Generalizing inner product and norm
  - Generalizing the matched filter (receiver frontend)
  - Adapting the signal space perspective
  - Generalizing the probability of error expressions

Inner Products and Norms

- The inner product between two complex signals $x(t)$ and $y(t)$ must be defined as

$$\langle x(t), y(t) \rangle = \int x(t) \cdot y^*(t)\, dt.$$

- This is needed to ensure that the resulting squared norm is positive and real

$$\|x(t)\|^2 = \langle x(t), x(t) \rangle = \int |x(t)|^2\, dt$$

Inner Products and Norms

- Norms are equal for passband and equivalent baseband signals.
  - Let

$$x_p(t) = \Re\{x(t)\sqrt{2}\exp(j2\pi f_c t)\}$$
$$y_p(t) = \Re\{y(t)\sqrt{2}\exp(j2\pi f_c t)\}$$

  - Then,

$$\langle x_p(t), y_p(t)\rangle = \Re\{\langle x(t), y(t)\rangle\}$$
$$= \langle x_I(t), y_I(t)\rangle + \langle x_Q(t), y_Q(t)\rangle$$

  - The first equation implies

$$\|x_P(t)\|^2 = \|x(t)\|^2$$

    - Remark: the factor $\sqrt{2}$ in $x_p(t) = \Re\{x(t)\sqrt{2}\exp(j2\pi f_c t)\}$ ensures this equality.

Receiver Frontend

- Let the baseband equivalent, received signal be $R(t) = R_I(t) + jR_Q(t)$.
- Then the optimum receiver frontend for the complex signal $s(t) = s_I(t) + js_Q(t)$ will compute

$$
\begin{aligned}
R &= \langle R_P(t), s_P(t) \rangle = \Re\{\langle R(t), s(t) \rangle\} \\
&= \langle R_I(t), s_I(t) \rangle + \langle R_Q(t), s_Q(t) \rangle
\end{aligned}
$$

  - The I and Q channel are first matched filtered individually and then added together.

Signal Space

- Assume that passband signals have the form

$$s_P(t) = b_I p(t)\sqrt{2E}\cos(2\pi f_c t) - b_Q p(t)\sqrt{2E}\sin(2\pi f_c t)$$

for $0 \leq t \leq T$.
  - where $p(t)$ is a unit energy pulse waveform.
- Orthonormal basis functions are

$$\Phi_0 = \sqrt{2}p(t)\cos(2\pi f_c t) \quad \text{and} \quad \Phi_1 = \sqrt{2}p(t)\sin(2\pi f_c t)$$

- The corresponding baseband signals are

$$s(t) = b_I p(t)\sqrt{E} + jb_Q p(t)\sqrt{E}$$

- with basis functions

$$\Phi_0 = p(t) \quad \text{and} \quad \Phi_1 = jp(t)$$

Probability of Error

- Expressions for the probability of error are unchanged as long as the above changes to inner product and norm are incorporated.
- Specifically, expressions involving the distance between signals are unchanged

$$Q\left(\frac{\|s_n - s_m\|}{\sqrt{2N_0}}\right).$$

- Expressions involving inner products with a suboptimal signal $g(t)$ are modified to

$$Q\left(\frac{\Re\{\langle s_n - s_m, g(t)\rangle\}}{\sqrt{2N_0}\|g(t)\|}\right)$$

Summary

- The baseband equivalent channel model is much simpler than the passband model.
  - Up and down conversion are eliminated.
  - Expressions for signals do not contain carrier terms.
- The baseband equivalent signals are more tractable and easier to model (e.g., for simulation).
  - Since they are low-pass signals, they are easily sampled.
- No information is lost when using baseband equivalent signals, instead of passband signals.
- Standard, linear system equations hold (nearly)
- **Conclusion:** Use baseband equivalent signals and systems.

Introduction

- For our discussion of optimal receivers, we have focused on
  - the transmission of single symbols and
  - the signal space properties of symbol constellations.
  - We recognized the critical importance of distance between constellation points.
- The precise shape of the transmitted waveforms plays a secondary role when it comes to error rates.
- However, the spectral properties of transmitted signals depends strongly on the shape of signals.