Introduction

- We compare methods for transmitting a sequence of bits.
- We will see that the performance of these methods varies significantly.
- **New perspective:**
  - Focus on messages, i.e., sequences of bits
  - Entire message must be received correctly
- Main Result: It is possible to achieve error free communications as long as SNR is good enough and data rate is not too high.

Problem Statement

- **Problem:**
  - K bits must be transmitted in T seconds.
  - Available power is limited to P.
- **Questions:**
  - What method achieves the lowest probability of error?
  - Is error-free communications possible?

Parameters

- **Data Rate:**
$$R = \frac{K}{T} \quad \text{(bits/s)}$$
  - entire transmission takes T seconds
  - K bits are sent over T seconds
  - implicit assumption: bits are equally likely.
- **Power and energy:** transmitted signal $s(t)$ has power P and energy E
$$P = \frac{1}{T} \int_0^T |s(t)|^2 \, dt = \frac{E}{T}$$
  - Entire transmitted signal $s(t)$ is of duration T.
  - Note, bit energy is given by
$$E_b = \frac{E}{K} = \frac{PT}{K} = \frac{P}{R}.$$

Bit-by-bit Signaling

- Transmit K bit as a sequence of “one-shot” BPSK signals.
- $K = RT$ bits to be transmitted.
- Energy per bit E_b ($E_b = \frac{E}{K}$).
- Consider, signals of the form

$$s(t) = \sum_{k=0}^{K-1} \sqrt{E_b} s_k p(t - k/R)$$

  - $s_k \in \{\pm 1\}$
  - $p(t)$ is a pulse of duration $1/R = T/K$ and $\|p(t)\|^2 = 1$.
- **Question:** What is the probability that any transmission error occurs?
  - In other words, the transmission is not received without error.

Error Probability for Bit-by-Bit Signaling

- We can consider the entire message as a single K-dimensional signal set.
  - Signals are at the vertices of a K-dimensional hypercube.

$$\Pr\{e\} = 1 - \left(1 - Q\left(\frac{2E_b}{N_0}\right)\right)^K$$
$$= 1 - \left(1 - Q\left(\frac{2P}{RN_0}\right)\right)^{RT}$$

- Note, for any finite P/N_0 and R, the error rate will always tend to 1 as $T \rightarrowtail \infty$.
  - Error-free transmission is *not* possible with bit-by-bit signaling.

Block-Orthogonal Signaling

- Again,
  - $K = RT$ bits are transmitted in T seconds.
  - Energy per bit $E_b = \frac{P}{R}$.
- Signal set (Pulse-position modulation — PPM)

$$s_k(t) = \sqrt{E}p(t - kT/2^K) \quad \text{for } k = 0, 1, \ldots, 2^K - 1.$$

where $p(t)$ is of duration $T/2^K$, $E = KE_b$, and $\|p(t)\|^2 = 1$.

- Alternative signal set (Frequency Shift Keying — FSK)

$$s_k(t) = \sqrt{\frac{2E}{T}} \cos(2\pi(f_c + k/T)t) \quad \text{for } k = 0, 1, \ldots, 2^K - 1.$$

- Signal set consists of $M = 2^K$ signals
  - each signal conveys K bits,
  - each signal occupies one of the K dimensions.

Union Bound

- The error probability for block-orthogonal signaling cannot be computed in closed form.
- At high and moderate SNR, the error probability is well approximated by the union bound.
  - Each signal has $M-1=2^K-1$ nearest neighbors.
  - The distance between neighbors is $d_{\text{min}}=\sqrt{2E}=\sqrt{2KE_b}$.
- Union bound

$$
\begin{aligned}
\Pr\{e\} &\leq (2^K-1)Q\left(\sqrt{\frac{KE_b}{N_0}}\right) \\
&= (2^{RT}-1)Q\left(\sqrt{\frac{PT}{N_0}}\right)
\end{aligned}
$$

Bounding the Union Bound

- To gain further insight, we bound

$$Q(x) \leq \frac{1}{2}\exp(-x^2/2) \leq \exp(-x^2/2).$$

- Then,

$$\begin{aligned}
\Pr\{e\} &\leq (2^{RT}-1)Q\left(\sqrt{\frac{PT}{N_0}}\right) \\
&\lesssim 2^{RT}\exp\left(-\frac{PT}{2N_0}\right) \\
&= \exp\left(-T\left(\frac{P}{2N_0}-R\ln 2\right)\right).
\end{aligned}$$

- Hence, $\Pr\{e\} \to 0$ as $T \to \infty$!
  - As long as $R < \frac{1}{\ln 2}\frac{P}{2N_0}$.
- Error-free transmission is possible!

Reality-Check: Bandwidth

- **Bit-by-bit Signaling:** Pulse-width: $T/K = 1/R$.
  - Bandwidth is approximately equal to $B = R$.
  - Also, number of dimensions $K = RT$.
- **Block-orthogonal:** Pulse width: $T/2^K = T/2^{RT}$.
  - Bandwidth is approximately equal to $B = 2^{RT}/T$.
  - Number of dimensions is $2^K = 2^{RT}$.
- Bandwidth for block-orthogonal signaling grows exponentially with the number of bits K.
  - Not practical for moderate to large blocks of bits.

The Dimensionality Theorem

- The relationship between bandwidth B and the number of dimensions is summarized by the *dimensionality theorem:*
  - The number of dimensions D available over an interval od duration T is limited by the bandwidth B

$$D \leq B \cdot T$$

  - The theorem implies:
    - A signal occupying D dimensions over T seconds requires bandwidth

$$B \geq \frac{D}{T}$$

An Ideal Signal Set

- An ideal signal set combines the aspects of our two example signal sets:
  - $\Pr\{e\}$-behavior like block orthogonal signaling

$$\lim_{T \to \infty} \Pr\{e\} = 0.$$

  - Bandwidth behavior like bit-by-bit signaling

$$B = \frac{D}{T} = \text{constant}.$$

    - Thus, $D = BT \to \infty$ as $T \to \infty$.
- **Question:** Does such a signal set exist?

Towards Channel Capacity

- **Given:**
  - bandwidth $B = \frac{D}{T}$, where T is the duration of the transmission.
  - power P
  - Noise power spectral density $\frac{N_0}{2}$
- **Question:** What is the highest data rate R that allows error-free transmission with the above constraints?
  - We are transmitting RT bits
  - Therefore, we need $M = 2^{RT}$ signals.

Signal Set

- Our signal set consists of $M = 2^{RT}$ signals of the form

$$s_n(t) = \sum_{k=0}^{D-1} X_{n,k} p(t - kT/D)$$

  where
  - $p(t)$ are pulses of duration T/D, i.e., of bandwidth $B = D/T$.
  - Also, $\|p(t)\|^2 = 1$.
- Each signal $s_n(t)$ is defined by a length-D vector $\vec{X}_n = \{X\}_{n,k}$.
- We are looking to find $M = 2^{RT}$ length-D vectors $\vec{X}$ that lead to good error properties.
- Note that the signals $p(t - kT/D)$ form an orthonormal basis with D dimensions.

Receiver Frontend

- The receiver frontend consists of a matched filter for $p(t)$ followed by a sampler at times kT/D.
  - I.e., the frontend projects the received signal onto the orthonormal basis functions $p(t - kT/D)$.
- The vector $\vec{R}$ of matched filter outputs has elements

$$R_k = \langle R_t, p(t - kT/D) \rangle \quad k = 0, 1, \ldots, D-1$$

- Conditional on $s_n(t)$ was sent, $\vec{R} \sim N(\vec{X}_n, \frac{N_o}{2} I)$.
- The optimum receiver selects the signal s_n that's closest to $\vec{R}$.

Conditional Error Probability

- When, the signal $s_n(t)$ was sent then $\vec{R} \sim N(\vec{X}_n, \frac{N_o}{2} I)$.
- As the number of dimensions D increases, the vector $\vec{R}$ lies within a D-dimensional sphere with center $\vec{X}_k$ and radius $\sqrt{D\frac{N_0}{2}}$ with very high probability: $1 - e^{-D}$, i.e., $P_e = e^{-D}$.
  - **Important:** We allow the radius of the decoding spheres to grow with the number of dimensions D.
  - This ensures that $P_e \to 0$ as $D = BT \to \infty$.
- We call the spheres of radius $\sqrt{D\frac{N_0}{2}}$ around each signal point *decoding spheres*.
  - The decoding spheres will be part of the decision regions for each point.

Power Constraint

- The power for signal $s_n(t)$ must satisfy

$$\frac{1}{T}\int_0^T s_n^2(t)\,dt = \frac{1}{T}\sum_{k=0}^{D-1}|X_{n,k}|^2 = \frac{1}{T}\|\vec{X}_n\|^2 \leq P.$$

- Therefore, $\|\vec{X}_n\|^2 \leq PT$
- Insights:
  - The transmitted signals lie in a sphere of radius $\sqrt{PT}$.
  - The observed signals must lie in a large sphere of radius $\sqrt{PT + D\frac{N_0}{2}}$.
- **Question:** How many decoding spheres can we have and still meet the power constraint?

Capacity

- Each decoding sphere has volume $K_D(\sqrt{D\frac{N_0}{2}})^D$.
- The volume of the sphere containing the observed signals is $K_D(\sqrt{PT + D\frac{N_0}{2}})^D$
  - K_D is a constant that depends only on the number of dimensions D, e.g., $K_3 = \frac{4\pi}{3}$.
- The number of decoding spheres that fit into the the power sphere is (upper) bounded by the ratio of the volumes

$$\frac{K_D\left(\sqrt{PT + D\frac{N_0}{2}}\right)^D}{K_D\left(\sqrt{D\frac{N_0}{2}}\right)^D}$$

Capacity

- Since the number of signals $M = 2^{RT}$ equals the number of decoding spheres, it follows that error free communications is possible (in the limit as $D = BT \to \infty$) if

$$M = 2^{RT} < \frac{\left(\sqrt{PT + D\frac{N_0}{2}}\right)^D}{\left(\sqrt{D\frac{N_0}{2}}\right)^D}$$

or

$$R < \frac{D}{2T} \log_2(1 + \frac{PT}{DN_0/2}) = \frac{B}{2} \log_2(1 + \frac{P}{BN_0/2}).$$

  - Note, if we allow *complex valued* signals, then $R < B \log_2(1 + \frac{P}{BN_0})$.

Illustration: 2-bit Messages

- Consider two different ways of transmitting two bits:
  - QPSK
  - rate $2/3$ block code and BPSK modulation
- Compare the probability of at least one bit error
  - constant $\frac{E_b}{N_0}$.

QPSK

- We know that for QPSK
  - energy efficiency $\eta_u = 4$
  - (symbol) error rate

$$P_e \leq 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

Benefit of a Simple Code

- The block code maps two bits to sequence of three BPSK symbols as follows:

$$00 : \{1, 1, 1\} \qquad 01 : \{1, -1, -1\}$$
$$10 : \{-1, 1, -1\} \qquad 11 : \{-1, -1, 1\}$$

- For this signal set:
  - energy efficiency $\eta_c = \frac{16}{3}$
  - (symbol) error rate

$$P_e \leq 3\mathrm{Q}\left(\sqrt{\frac{8E_b}{3N_0}}\right)$$

- Coding gain:

$$\frac{\eta_c}{\eta_u} = \frac{16/3}{4} = \frac{4}{3} \approx 1\text{dB}$$