ECE 201: Introduction to Signal Analysis

Prof. Paris

Last updated: October 9, 2007

Part I

Spectrum Representation of Signals

Lecture: Sums of Sinusoids (of different frequency)

Introduction

- To this point we have focused on sinusoids of identical frequency f

$$x(t) = \sum_{i=1}^{N} A_i \cos(2\pi f t + \phi_i).$$

  - Note that the frequency f does not have a subscript i!
- Showed (in phasor addition rule) that the above sum can always be written as a single sinusoid of frequency f.

Introduction

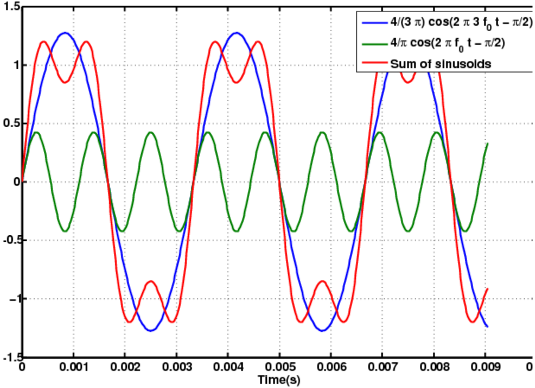
- We will consider sums of sinusoids of different frequencies:

$$x(t) = \sum_{i=1}^{N} A_i \cos(2\pi f_i t + \phi_i).$$

  - Note the subscript on the frequencies f_i!
  - This apparently minor difference has dramatic consequences.

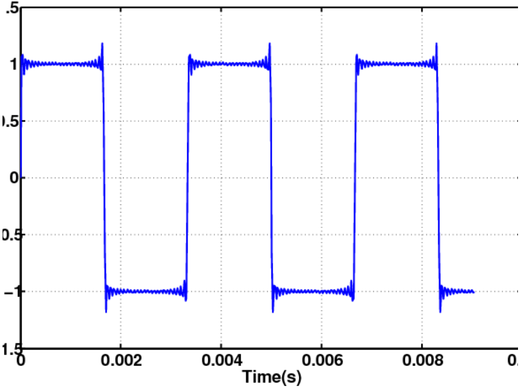
Sum of Two Sinusoids

$$x(t) = \frac{4}{\pi}\cos(2\pi ft - \pi/2) + \frac{4}{3\pi}\cos(2\pi 3ft - \pi/2)$$



Sum of 25 Sinusoids

$$x(t) = \sum_{n=0}^{25} \frac{4}{(2n-1)\pi} \cos(2\pi(2n-1)ft - \pi/2)$$



Non-sinusoidal Signals as Sums of Sinusoids

- If we allow infinitely many sinusoids in the sum, then the result is a square wave signal.
- The example demonstrates that general, non-sinusoidal signals can be represented as a sum of sinusoids.
  - The sinusods in the summation depend on the general signal to be represented.
  - For the square wave signal we need sinusoids
    - of frequencies $(2n-1) \cdot f$, and
    - amplitudes $\frac{4}{(2n-1)\pi}$.
    - (This is not obvious).

Non-sinusoidal Signals as Sums of Sinusoids

- The ability to express general signals in terms of sinusoids forms the basis for the frequency domain or spectrum representation.
    - **Basic idea:** list the *"ingredients"* of a signal by specifying amplitudes and phases as well as frequencies of the sinusoids in the sum.

The Spectrum of a Sum of Sinusoids

- Begin with the sum of sinusoids introduced earlier

$$x(t) = A_0 + \sum_{i=1}^{N} A_i \cos(2\pi f_i t + \phi_i).$$

  where we have broken out a possible constant term.
  - The term A_0 can be thought of as corresponding to a sinusoid of frequency zero.
- Using the *inverse Euler formula*, we can replace the sinusoids by complex exponentials

$$x(t) = X_0 + \sum_{i=1}^{N} \left\{ \frac{X_i}{2} \exp(j2\pi f_i t) + \frac{X_i^*}{2} \exp(-j2\pi f_i t) \right\}.$$

  where $X_0 = A_0$ and $X_i = A_i e^{j\phi_i}$.

The Spectrum of a Sum of Sinusoids (cont'd)

- Starting with

$$x(t) = X_0 + \sum_{i=1}^{N} \left\{ \frac{X_i}{2} \exp(j2\pi f_i t) + \frac{X_i^*}{2} \exp(-j2\pi f_i t) \right\}.$$

where $X_0 = A_0$ and $X_i = A_i e^{j\phi_i}$.

- The spectrum representation simply lists the complex amplitudes and frequencies in the summation:

$$X(f) = \{(X_0, 0), (\frac{1}{2}X_1, f_1), (\frac{1}{2}X_1^*, -f_1), \ldots, (\frac{1}{2}X_N, f_N), (\frac{1}{2}X_N^*, -f_N)\}$$

Example

- Consider the signal

$$x(t) = 3 + 5\cos(20\pi t - \pi/2) + 7\cos(50\pi t + \pi/4).$$

- Using the inverse Euler relationship

$$\begin{aligned} x(t) &= 3 + \quad \tfrac{5}{2}e^{-j\pi/2}\exp(j2\pi 10t) + \tfrac{5}{2}e^{j\pi/2}\exp(-j2\pi 10t) + \\ &\quad\quad \tfrac{7}{2}e^{j\pi/4}\exp(j2\pi 25t) + \tfrac{7}{2}e^{-j\pi/4}\exp(-j2\pi 25t). \end{aligned}$$

- Hence,

$$\begin{aligned} X(f) = \{(3,0), \quad & (\tfrac{5}{2}e^{-j\pi/2}, 10), (\tfrac{5}{2}e^{j\pi/2}, -10), \\ & (\tfrac{7}{2}e^{j\pi/4}, 25), (\tfrac{7}{2}e^{-j\pi/4}, -25)\} \end{aligned}$$

Exercise

- Find the spectrum of the signal:

$$x(t) = 6 + 4\cos(10\pi t + \pi/3) + 5\cos(20\pi t - \pi/7).$$

Lecture: From Time-Domain to Frequency-Domain and back

Time-domain and Frequency-domain

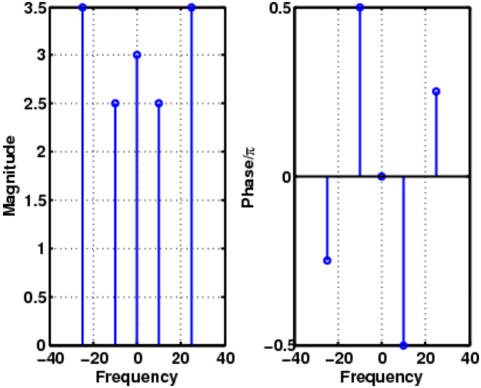
- Signals are *naturally* observed in the time-domain.
- A signal can be illustrated in the time-domain by plotting it as a function of time.
- The frequency-domain provides an alternative perspective of the signal based on sinusoids:
    - Starting point: arbitrary signals can be expressed as sums of sinusoids (or equivalently complex exponentials).
    - The frequency-domain representation of a signal indicates which complex exponentials must be combined to produce the signal.
    - Since complex exponentials are fully described by amplitude, phase, and frequency it is sufficient to just specify a list of theses parameters.
        - Actually, we list pairs of complex amplitudes ($Ae^{j\phi}$) and frequencies f and refer to this list as $X(f)$.

Time-domain and Frequency-domain

- It is possible (but not necessarily easy) to find $X(f)$ from $x(t)$: this is called Fourier or spectrum analysis.
- Similarly, one can construct $x(t)$ from the spectrum $X(f)$: this is called Fourier synthesis.
- Notation: $x(t) \leftrightarrow X(f)$.
- Example (from last time):
  - **Time-domain:** signal
    $$x(t) = 3 + 5\cos(20\pi t - \pi/2) + 7\cos(50\pi t + \pi/4).$$
  - **Frequency Domain:** spectrum
    $$X(f) = \{(3,0), \quad \begin{array}{l}(\tfrac{5}{2}e^{-j\pi/2}, 10), (\tfrac{5}{2}e^{j\pi/2}, -10), \\ (\tfrac{7}{2}e^{j\pi/4}, 25), (\tfrac{7}{2}e^{-j\pi/4}, -25)\}\end{array}$$

Plotting a Spectrum

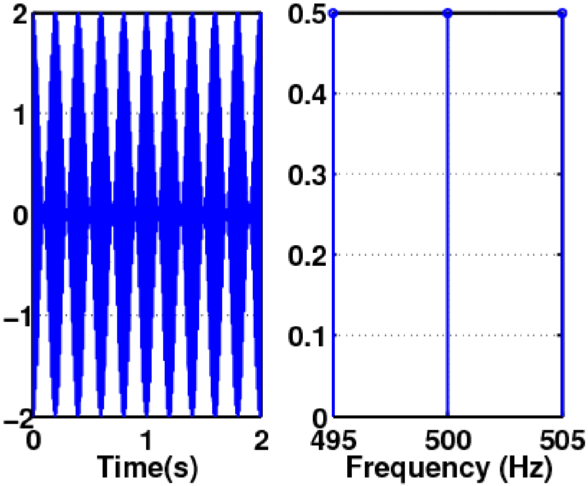
- To illustrate the spectrum of a signal, one typically plots the magnitude versus frequency.
- Sometimes the phase is plotted versus frequency as well.



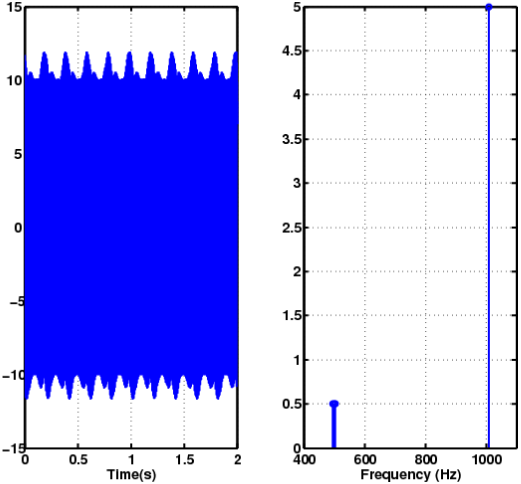
Why Bother with the Frequency-Domain?

- In many applications, the frequency contents of a signal is very important.
    - For example, in radio communications signals must be limited to occupy only a set of frequencies allocated by the FCC.
    - Hence, understanding and analyzing the spectrum of a signal is crucial from a regulatory perspective.
- Often, features of a signal are much easier to understand in the frequency domain. (Example on next slides).
- We will see later in this class, that the frequency-domain interpretation of signals is very useful in connection with linear, time-invariant systems.
    - Example: A low-pass filter retains low frequency components of the spectrum and removes high-frequency components.

Example: Original signal



Example: Corrupted signal



Synthesis: From Frequency to Time-Domain

- Synthesis is a straightforward process; it is a lot like following a recipe.
- *Ingredients* are given by the spectrum

  $$X(f) = \{(X_0, 0), (X_1, f_1), (X_1^*, -f_1), \ldots, (X_N, f_N), (X_N^*, -f_N)\}$$

  Each pair indicates one complex exponential component by listing its frequency and complex amplitude.
- *Instructions* for combining the ingredients and producing the (time-domain) signal:

  $$x(t) = \sum_{n=-N}^{N} X_n \exp(j2\pi f_n t).$$

- You should simplify the expression you obtain.

Example

- Problem: Find the signal $x(t)$ corresponding to

$$\begin{aligned} X(f) = \{(3,0), \quad & (\tfrac{5}{2}e^{-j\pi/2}, 10), (\tfrac{5}{2}e^{j\pi/2}, -10), \\ & (\tfrac{7}{2}e^{j\pi/4}, 25), (\tfrac{7}{2}e^{-j\pi/4}, -25)\} \end{aligned}$$

- Solution:

$$\begin{aligned} x(t) = 3 \quad & + \tfrac{5}{2}e^{-j\pi/2}e^{j2\pi 10t} + \tfrac{5}{2}e^{j\pi/2}e^{-j2\pi 10t} \\ & + \tfrac{7}{2}e^{j\pi/4}e^{j2\pi 25t} + \tfrac{7}{2}e^{-j\pi/4}e^{-j2\pi 25t} \end{aligned}$$

- Which simplifies to:

$$x(t) = 3 + 5\cos(20\pi t - \pi/2) + 7\cos(50\pi t + \pi/4).$$

Exercise

- Find the signal with the spectrum:

$$X(f) = \{(5,0), \quad \begin{array}{l}(2e^{-j\pi/4}, 10), (2e^{j\pi/4}, -10), \\ (\frac{5}{2}e^{j\pi/4}, 15), (\frac{5}{2}e^{-j\pi/4}, -15)\end{array}$$

Analysis: From Time to Frequency-Domain

- The objective of spectrum or Fourier analysis is to find the spectrum of a time-domain signal.
- We will restrict ourselves to signals $x(t)$ that are sums of sinusoids

$$x(t) = A_0 + \sum_{i=1}^{N} A_i \cos(2\pi f_i t + \phi_i).$$

- We have already shown that such signals have spectrum:

$$X(f) = \{(X_0, 0), (\frac{1}{2}X_1, f_1), (\frac{1}{2}X_1^*, -f_1), \ldots, (\frac{1}{2}X_N, f_N), (\frac{1}{2}X_N^*, -f_N)\}$$

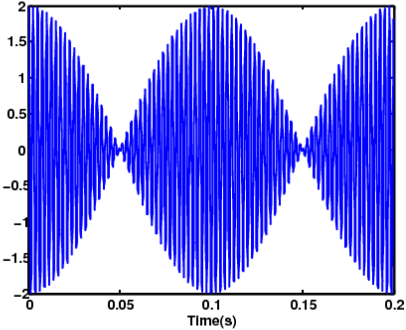
  where $X_0 = A_0$ and $X_i = A_i e^{j\phi_i}$.
- We will investigate some interesting signals that can be written as a sum of sinusoids.

Beat Notes

- Consider the signal

$$x(t) = 2 \cdot \cos(2\pi 5t) \cdot \cos(2\pi 400t).$$

- This signal does not have the form of a sum of sinusoids; hence, we can not determine it's spectrum immediately.



MATLAB Code for Beat Notes

```
% BeatNote - plot and play a beat note waveform

% Parameters
fs  = 8192;
dur = 2;
NP  = round(fs/5);

f1 = 5;
f2 = 400;
A  = 2;

% time axis
tt=0:1/fs:dur;

xx = A*cos(2*pi*f1*tt).*cos(2*pi*f2*tt);

plot(tt(1:NP),xx(1:NP))
xlabel('Time(s)')

soundsc(xx,fs);
```

Beat Notes as a Sum of Sinusoids

- Using the inverse Euler relationships, we can write

$$\begin{aligned} x(t) &= 2 \cdot \cos(2\pi 5t) \cdot \cos(2\pi 400t) \\ &= 2 \cdot \tfrac{1}{2} \cdot (e^{j2\pi 5t} + e^{-j2\pi 5t}) \cdot \tfrac{1}{2} \cdot (e^{j2\pi 400t} + e^{-j2\pi 400t}). \end{aligned}$$

- Multiplying out yields:

$$x(t) = \frac{1}{2}(e^{j2\pi 405t} + e^{-j2\pi 405t}) + \frac{1}{2}(e^{j2\pi 395t} + e^{-j2\pi 395t}).$$

- Applying Euler's relationship, lets us write:

$$x(t) = \cos(2\pi 405t) + \cos(2\pi 395t).$$

Spectrum of Beat Notes

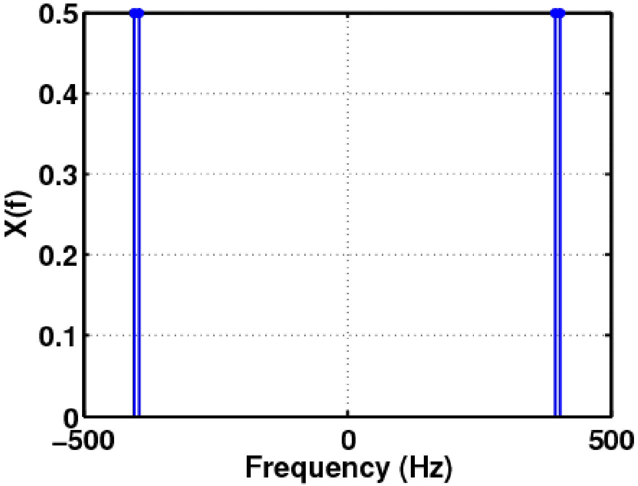
- We were able to rewrite the beat notes as a sum of sinusoids

$$x(t) = \cos(2\pi 405t) + \cos(2\pi 395t).$$

- Note that the frequencies in the sum, 395 Hz and 405 Hz, are the sum and difference of the frequencies in the original product, 5 Hz and 400 Hz.
- It is now straightforward to determine the spectrum of the beat notes signal:

$$X(f) = \{(\frac{1}{2}, 405), (\frac{1}{2}, -405), (\frac{1}{2}, 395), (\frac{1}{2}, -395)\}$$

Spectrum of Beat Notes



Lecture: Amplitude Modulation and Periodic Signals

Amplitude Modulation

- Amplitude Modulation is used in communication systems.
- The objective of amplitude modulation is to move the spectrum of a signal $m(t)$ from low frequencies to high frequencies.
    - The message signal $m(t)$ may be a piece of music; its spectrum occupies frequencies below 20 KHz.
    - For transmission by an AM radio station this spectrum must be moved to approximately 1 MHz.

Amplitude Modulation

- Conventional amplitude modulation proceeds in two steps:
  - A constant A is added to $m(t)$ such that $A + m(t) > 0$ for all t.
  - The sum signal $A + m(t)$ is multiplied by a sinusoid $\cos(2\pi f_c t)$, where f_c is the radio frequency assigned to the station.
- Consequently, the transmitted signal has the form:

$$x(t) = (A + m(t)) \cdot \cos(2\pi f_c t).$$

Amplitude Modulation

- We are interested in the spectrum of the AM signal.
- However, we cannot compute $X(f)$ for arbitrary message signals $m(t)$.
- For the special case $m(t) = \cos(2\pi f_m t)$ we can find the spectrum.
  - To mimic the radio case, f_m would be a frequency in the audible range.
- As before, we will first need to express the AM signal $x(t)$ as a sum of sinusoids.

Amplitude Modulated Signal

- For $m(t) = \cos(2\pi f_m t)$, the AM signal equals

$$x(t) = (A + \cos(2\pi f_m t)) \cdot \cos(2\pi f_c t).$$

- This simplifies to

$$x(t) = A \cdot \cos(2\pi f_c t) + \cos(2\pi f_m t) \cdot \cos(2\pi f_c t).$$

- Note that the second term of the sum is a beat notes signal with frequencies f_m and f_c.
- We know that beat notes can be written as a sum of sinusoids with frequencies equal to the sum and difference of f_m and f_c:

$$x(t) = A \cdot \cos(2\pi f_c t) + \frac{1}{2}\cos(2\pi(f_c + f_m)t) + \frac{1}{2}\cos(2\pi(f_c - f_m)t).$$

Spectrum of Amplitude Modulated Signal

- The AM signal is given by

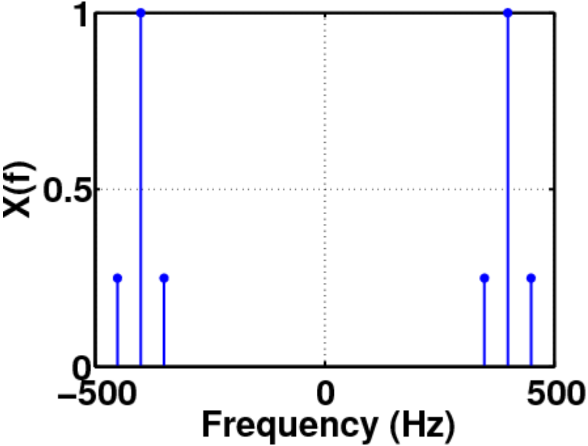
$$x(t) = A \cdot \cos(2\pi f_c t) + \frac{1}{2}\cos(2\pi(f_c + f_m)t) + \frac{1}{2}\cos(2\pi(f_c - f_m)t).$$

- Thus, its spectrum is

$$\begin{aligned} X(f) = \{ \quad & (\tfrac{A}{2}, f_c), (\tfrac{A}{2}, -f_c), \\ & (\tfrac{1}{4}, f_c + f_m), (\tfrac{1}{4}, -f_c - f_m), (\tfrac{1}{4}, f_c - f_m), (\tfrac{1}{2}, -f_c + f_m)\} \end{aligned}$$

Spectrum of Amplitude Modulated Signal

For $A = 2$, $fm = 50$, and $fc = 400$, the spectrum of the AM signal is plotted below.



Spectrum of Amplitude Modulated Signal

- It is interesting to compare the spectrum of the signal before modulation and after multiplication with $\cos(2\pi f_c t)$.
- The signal $s(t) = A + m(t)$ has spectrum

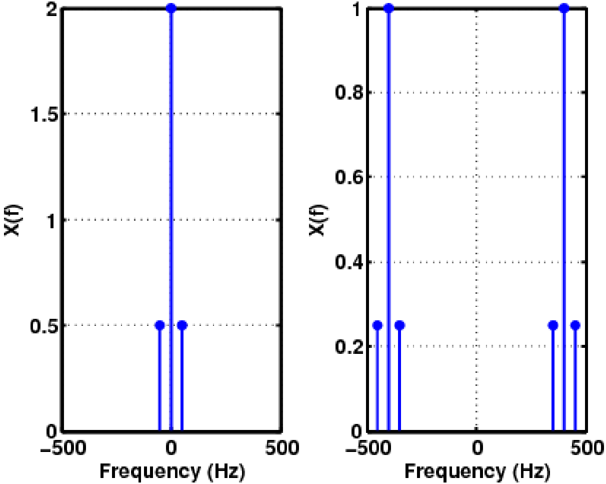
$$S(f) = \{(A, 0), (\frac{1}{2}, 50), (\frac{1}{2}, -50)\}.$$

- The modulated signal $x(t)$ has spectrum

$$\begin{aligned} X(f) = \{ \quad & (\tfrac{A}{2}, 400), (\tfrac{A}{2}, -400), \\ & (\tfrac{1}{4}, 450), (\tfrac{1}{4}, -450), (\tfrac{1}{4}, 350), (\tfrac{1}{2}, -350)\} \end{aligned}$$

- Both are plotted on the next page.

Spectrum before and after AM



Spectrum before and after AM

- Comparison of the two spectra shows that amplitude module indeed moves a spectrum from low frequencies to high frequencies.
- Note that the shape of the spectrum is precisely preserved.
- Amplitude modulation can be described concisely by stating:
  - Half of the original spectrum is shifted by f_c to the right, and the other half is shifted by f_c to the left.
- **Question:** How can you get the original signal back so that you can listen to it.
  - This is called demodulation.

What are Periodic Signals?

- A signal $x(t)$ is called periodic if there is a constant T_0 such that
$$x(t) = x(t + T_0) \text{ for all } t.$$
- In other words, a periodic signal repeats itself every T_0 seconds.
- The interval T_0 is called the fundamental period of the signal.
- The inverse of T_0 is the fundamental frequency of the signal.
- Example:
  - A sinusoidal signal of frequency f is periodic with period $T_0 = 1/f$.

Harmonic Frequencies

- Consider a sum of sinusoids:

$$x(t) = A_0 + \sum_{i=1}^{N} A_i \cos(2\pi f_i t + \phi_i).$$

- A special case arises when we constrain all frequencies f_i to be integer multiples of some frequency f_0:

$$f_i = i \cdot f_0.$$

- The frequencies f_i are then called harmonic frequencies of f_0.
- We will show that sums of sinusoids with frequencies that are harmonics are periodic.

Harmonic Signals are Periodic

- To establish periodicity, we must show that there is T_0 such $x(t) = x(t + T_0)$.
- Begin with

$$\begin{aligned} x(t + T_0) &= A_0 + \sum_{i=1}^{N} A_i \cos(2\pi f_i(t + T_0) + \phi_i) \\ &= A_0 + \sum_{i=1}^{N} A_i \cos(2\pi f_i t + 2\pi f_i T_0 + \phi_i) \end{aligned}$$

- Now, let $f_0 = 1/T_0$ and use the fact that frequencies are harmonics: $f_i = i \cdot f_0$.

Harmonic Signals are Periodic

- Then, $f_i \cdot T_0 = i \cdot f_0 \cdot T_0 = i$ and hence

$$\begin{aligned} x(t+T_0) &= A_0 + \sum_{i=1}^{N} A_i \cos(2\pi f_i t + 2\pi f_i T_0 + \phi_i) \\ &= A_0 + \sum_{i=1}^{N} A_i \cos(2\pi f_i t + 2\pi i + \phi_i) \end{aligned}$$

- We can drop the $2\pi i$ terms and conclude that $x(t+T_0) = x(t)$.
- **Conclusion:** A signal of the form

$$x(t) = A_0 + \sum_{i=1}^{N} A_i \cos(2\pi i \cdot f_0 t + \phi_i)$$

is periodic with period $T_0 = 1/f_0$.

Finding the Fundamental Frequency

- Often one is given a set of frequencies $f_1, f_2, \ldots, f_N$ and is required to find the fundamental frequency f_0.
- Specifically, this means one must find a frequency f_0 and integers $n_1, n_2, \ldots, n_N$ such that all of the following equations are met:

$$\begin{aligned} f_1 &= n_1 \cdot f_0 \\ f_2 &= n_2 \cdot f_0 \\ &\vdots \\ f_N &= n_N \cdot f_0 \end{aligned}$$

- Note that there isn't always a solution to the above problem.
  - However, if all frequencies are integers a solution exists.
  - Even if all frequencies are rational a solution exists.

Example

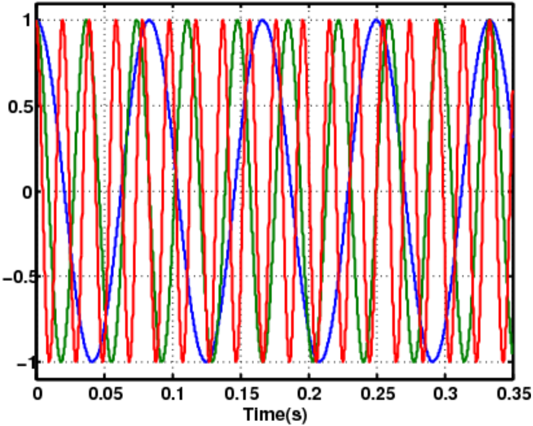
- Find the fundamental frequency for the set of frequencies $f_1 = 12, f_2 = 27, f_3 = 51$.
- Set up the equations:

$$\begin{aligned} 12 &= n_1 \cdot f_0 \\ 27 &= n_2 \cdot f_0 \\ 51 &= n_3 \cdot f_0 \end{aligned}$$

- Try the solution $n_1 = 1$; this would imply $f_0 = 12$. This cannot satisfy the other two equations.
- Try the solution $n_1 = 2$; this would imply $f_0 = 6$. This cannot satisfy the other two equations.
- Try the solution $n_1 = 3$; this would imply $f_0 = 4$. This cannot satisfy the other two equations.
- Try the solution $n_1 = 4$; this would imply $f_0 = 3$. This **can** satisfy the other two equations with $n_2 = 9$ and $n_3 = 17$.

Example

- Note that the three sinusoids complete a cycle at the same time at $T_0 = 1/f_0 = 1/3s$.



Exercise

- Find the fundamental frequency for the set of frequencies $f_1 = 2, f_2 = 3.5, f_3 = 5$.

Fourier Series

- We have shown that a sum of sinusoids with harmonic frequencies is a periodic signal.
- One can turn this statement around and arrive at a very important result:

> *Any periodic signal can be expressed as a sum of sinusoids with harmonic frequencies.*

- The resulting sum is called the Fourier Series of the signal.
- Put differently, a periodic signal can always be written in the form

$$\begin{aligned} x(t) &= A_0 + \sum_{i=1}^{N} A_i \cos(2\pi i f_0 t + \phi_i) \\ &= X_0 + \sum_{i=1}^{N} X_i e^{j2\pi i f_0 t} + X_i^* e^{-j2\pi i f_0 t} \end{aligned}$$

with $X_0 = A_0$ and $X_i = \frac{A_i}{2} e^{j\phi_i}$.

Fourier Series

- For a periodic signal the complex amplitudes X_i can be computed using a (relatively) simple formula.
- Specifically, for a periodic signal $x(t)$ with fundamental period T_0 the complex amplitudes X_i are given by:

$$X_i = \frac{1}{T_0} \int_0^{T_0} x(t) \cdot e^{-j2\pi it/T_0} dt.$$

  - Note that the integral above can be evaluated over any interval of length T_0.

Example: Square Wave

- A square wave signal

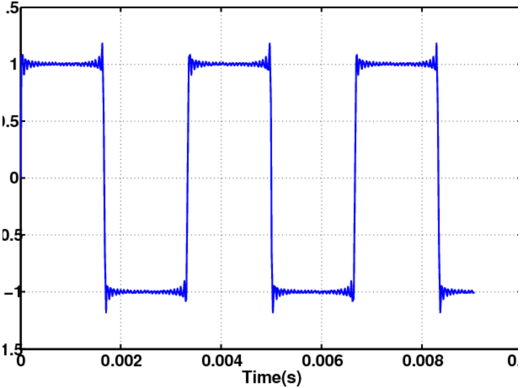
$$x(t) = \begin{cases} 1 & 0 \leq t < \frac{T_0}{2} \\ -1 & \frac{T_0}{2} \leq t < T_0 \end{cases}$$

can be written as

$$x(t) = \sum_{n=0}^{\infty} \frac{4}{(2n-1)\pi} \cos(2\pi(2n-1)ft - \pi/2)$$

25-Term Approximation to Square Wave

$$x(t) = \sum_{n=0}^{25} \frac{4}{(2n-1)\pi} \cos(2\pi(2n-1)ft - \pi/2)$$



Lecture: Time-Frequency Spectrum

Limitations of Sum-of-Sinusoid Signals

- So far, we have considered only signals that can be written as a sum of sinusoids.

$$x(t) = A_0 + \sum_{i=1}^{N} A_i \cos(2\pi f_i t + \phi_i).$$

- For such signals, we are able to compute the spectrum.
- Note, that signals of this form
  - are assumed to last forever, i.e., for $-\infty < t < \infty$,
  - and their spectrum never changes.
- While such signals are important and useful conceptually, they don't describe real-world signals accurately.
- Real-world signals
  - are of finite duration,
  - their spectrum changes over time.

Musical Notation

- Musical notation ("sheet music") provides a way to represent real-world signals: a piece of music.
- As you know, sheet music
  - places notes on a scale to reflect the *frequency* of the tone to be played,
  - uses differently shaped note symbols to indicate the *duration* of each tone,
  - provides the order in which notes are to be played.
- In summary, musical notation captures how the spectrum of the music-signal changes over time.
- We cannot write signals whose spectrum changes with time as a sum of sinusoids.
  - A *static* spectrum is insufficient to describe such signals.
- Alternative: time-frequency spectrum

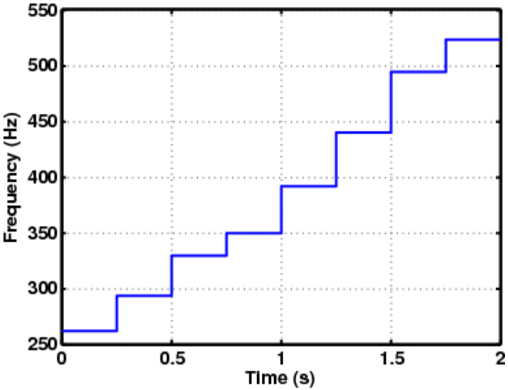
Example: Musical Scale

Note	C	D	E	F	G	A	B	C
Frequency (Hz)	262	294	330	349	392	440	494	523

Table: Musical Notes and their Frequencies

Example: Musical Scale

- If we play each of the notes for 250 ms, then the resulting signal can be summarized in the time-frequency spectrum below.

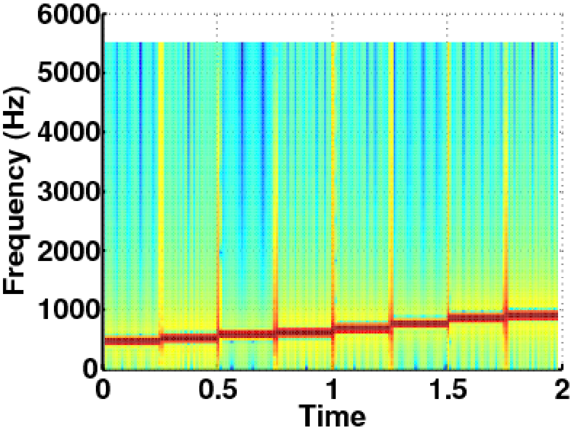


MATLAB Spectrogram Function

- MATLAB has a function `spectrogram` that can be used to compute the time-frequency spectrum for a given signal.
    - The resulting plots are similar to the one for the musical scale on the previous slide.
- Typically, you invoke this function as
  `spectrogram( xx, 256, 128, 256, fs)`,
  where `xx` is the signal to be analyzed and `fs` is the sampling frequency.
- The spectrogram for the musical scale is shown on the next slide.

Spectrogram: Musical Scale

- The color indicates the magnitude of the spectrum at a given time and frequency.



Chirp Signals

- **Objective:** construct a signal such that its frequency increases with time.
- **Starting Point:** A sinusoidal signal has the form:

$$x(t) = A\cos(2\pi f_0 t + \phi).$$

- We can consider the argument of the cos as a time-varying phase function

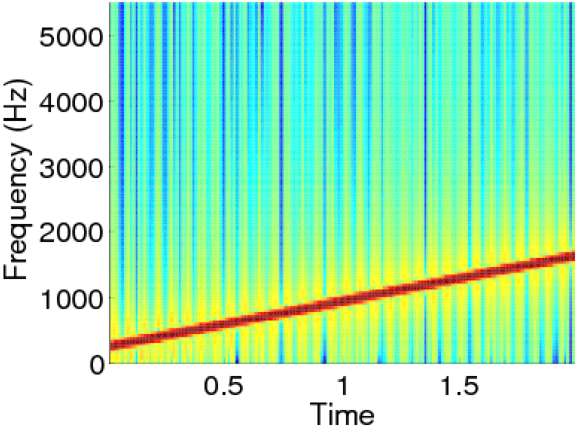
$$\Psi(t) = 2\pi f_0 t + \phi.$$

- **Question:** What happens when we allow more general functions for $\Psi(t)$?
  - For example, let

$$\Psi(t) = 700\pi t^2 + 440\pi t + \phi.$$

Spectrogram: $\cos(\Psi(t))$

- **Question:** How is he time-frequency spectrum related to $\Psi(t)$?



Instantaneous Frequency

- For a regular sinusoid, $\Psi(t) = 2\pi f_0 t + \phi$ and the frequency equals f_0.
- This suggests as a possible relationship between $\Psi(t)$ and f_0

$$f_0 = \frac{1}{2\pi}\frac{d}{dt}\Psi(t).$$

- If the above derivative is not a constant, it is called the instantaneous frequency of the signal, $f_i(t)$.
- **Example:** For $\Psi(t) = 700\pi t^2 + 440\pi t + \phi$ we find

$$f_i(t) = \frac{1}{2\pi}\frac{d}{dt}(700\pi t^2 + 440\pi t + \phi) = 700t + 220.$$

  - This describes precisely the red line in the spectrogram on the previous slide.

Constructing a Linear Chirp

- **Objective:** Construct a signal such that its frequency is initially f_1 and increases linear to f_2 after T seconds.
- **Solution:** The above suggests that

$$f_i(t) = \frac{f_2 - f_1}{T} t + f_1 .$$

- Consequently, the phase function $\Psi(t)$ must be

$$\Psi(t) = 2\pi \frac{f_2 - f_1}{2T} t^2 + 2\pi f_1 t + \phi$$

  - Note that ϕ has no influence on the spectrum; it is usually set to 0.

Constructing a Linear Chirp

- **Example:** Construct a linear chirp such that the frequency decreases from 1000 Hz to 200 Hz in 2 seconds.
- The desired signal must be

$$x(t) = \cos(-2\pi 200 t^2 + 2\pi 1000 t).$$

Exercise

- Construct a linear chirp such that the frequency increases from 50 Hz to 200 Hz in 3 seconds.
- Sketch the time-frequency spectrum of the following signal

$$x(t) = \cos(2\pi 500t + 100\cos(2\pi 2t))$$