**Problems**

1. **Hypothesis Testing with Laplacian Noise**
A random variable $N$ is said to be Laplacian distributed if its probability density function is given by

$$p_N(x) = \frac{1}{2}e^{-|x|} \quad \text{for } -\infty < x < \infty.$$

Consider the following decision problem involving the observed random variable $Z$:

$$H_0\colon Z = 2 + N$$
$$H_1\colon Z = -1 + N,$$

where the two hypotheses are equally likely.

(a) Provide expressions for the probability density function for $Z$ for each of the two hypotheses.

(b) Show that the maximum likelihood decision rule can be simplified to

$$Z \underset{H_1}{\overset{H_0}{\gtrless}} \gamma.$$

Determine the value of the optimum threshold $\gamma$.

(c) Compute the probability of error for this decision rule.

For the remainder of the problem, consider a two-dimensional random vector $\vec{N} = \binom{N_1}{N_2}$ with *independent* Laplacian distributed components, i.e.,

$$p_{\vec{N}}(\vec{x}) = p_{N_1}(x_1)p_{N_2}(x_2) = \frac{1}{4}e^{-(|x_1|+|x_2|)} \quad \text{for } -\infty < x_i < \infty.$$

with $\vec{x} = \binom{x_1}{x_2}$.
Consider the following decision problem involving the observed random vector $\vec{Z} = \binom{Z_1}{Z_2}$:

$$H_0\colon \vec{Z} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \vec{N}$$
$$H_1\colon \vec{Z} = \begin{pmatrix} -1 \\ -3 \end{pmatrix} + \vec{N},$$

where again the two hypotheses are equally likely.

(d) Provide expressions for the probability density function for $\vec{Z}$ for each of the two hypotheses.

(e) Show that the maximum likelihood decision rule can be simplified to

$$|Z_1 + 1| - |Z_1 - 2| \underset{H_1}{\overset{H_0}{\gtrless}} |Z_2 - 1| - |Z_2 + 3|.$$

(f) For each of the following observations, indicate if the decision rule decides $H_0$ or $H_1$:

i. $\vec{Z} = \binom{1.75}{-2}$

ii. $\vec{Z} = \binom{1.75}{-2.75}$

iii. $\vec{Z} = \binom{2.25}{-2.75}$

iv. $\vec{Z} = \binom{5}{-2.75}$

v. $\vec{Z} = \binom{5}{-3.25}$

(g) The absolute values in the decision rule induce three distinct intervals for $Z_1$ ($Z_1 < -1$, $-1 \leq Z_1 \leq 2$, and $Z_1 > 2$) and three intervals for $Z_2$ ($Z_2 < -3$, $-3 \leq Z_2 \leq 1$, and $Z_2 > 1$).
Consider all *nine* regions formed by combinations of these intervals (e.g., the region with $Z_1 < -1$ and $Z_2 < -3$) and simplify the decision rule for each of these combinations.

(h) Draw a two-dimensional signal-space diagram with axes $Z_1$ and $Z_2$. Mark the locations of $\mathbf{E}[\vec{Z}|H_i]$ for the two hypotheses. Then, draw the decision boundary formed by the optimal decision rule using the results from part (g).

a) $H_0: P_{Z|H_0}(z) = \frac{1}{2} e^{-|z-2|}$

$H_1: P_{Z|H_1}(z) = \frac{1}{2} e^{-|z+1|}$

b) LLR: $\ln\left(\frac{P_{Z|H_1}(z)}{P_{Z|H_0}(z)}\right) \underset{H_0}{\overset{H_1}{\gtrless}} \ln\frac{\pi_0}{\pi_1} = \ln(1) = 0$

$\Rightarrow \underline{-|z-2|} + |z+1| \underset{H_\phi}{\overset{H_0}{\gtrless}} 0$

3 cases summarized by:

$z \underset{H_1}{\overset{H_0}{\gtrless}} \frac{1}{2}$

3 cases:

a) $z \le -1$: $+(z-2)-(z+1) = -3$ $\Rightarrow H_\phi$

b) $-1 \le z \le 2$:

$(z-2)+(z+1) = 2z-1$

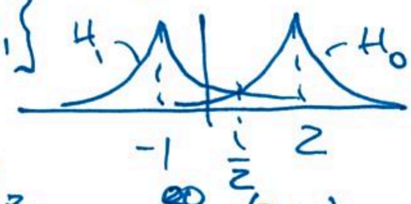
$2z-1 \underset{H_0}{\overset{H_1}{\gtrless}} 0$

$\Rightarrow z \underset{H_\phi}{\overset{H_0}{\gtrless}} \frac{1}{2}$

c) $z \ge 2$: $-(z-2)+(z+1) = 3$ $\Rightarrow H_1$

c) $P_e = \frac{1}{2}\Pr\{z \ge \frac{1}{2} | H_1\} + \frac{1}{2}\Pr\{z \le \frac{1}{2} | H_0\}$

$= \Pr\{z \ge \frac{1}{2} | H_1\}$




$= \int_{+\frac{1}{2}}^{\infty} \frac{1}{2} e^{-|z+1|}\, dz = \int_{\frac{1}{2}}^{\infty} \frac{1}{2} e^{-(z+1)}\, dz = \frac{1}{2} e^{-1} \cdot \int_{1/2}^{\infty} e^{-z}\, dz = \frac{1}{2} e^{-1} \cdot \left(-e^{-z}\right)\Big|_{1/2}^{\infty}$

$= \frac{1}{2} e^{-1} \cdot e^{-1/2} = \underline{\underline{\frac{1}{2} e^{-1.5}}}$

d) $P_{\underline{z}|H_0}(\underline{z}) = \frac{1}{4} \cdot e^{-|z_1-2|} \cdot e^{-|z_2-1|}$

$P_{\underline{z}|H_1}(\underline{z}) = \frac{1}{4} e^{-|z_1+1|} \cdot e^{-|z_2+3|}$

e) $\ln \frac{P_{\underline{z}|H_0}(\underline{z})}{P_{\underline{z}|H_1}(\underline{z})} = -|z_1-2| - |z_2-1| + |z_1+1| + |z_2+3| \underset{H_1}{\overset{H_0}{\gtrless}} 0$

$\Leftrightarrow |z_1+1| - |z_1-2| \underset{H_1}{\overset{H_0}{\gtrless}} |z_2-1| - |z_2+3|$

f) i) $2.75 - 0.25 \underset{H_1}{\overset{H_0}{\gtrless}} 3 - 1 \Rightarrow H_0$

ii) $2.75 - 0.25 \gtrless 3.75 - 0.25 \Rightarrow H_1$

iii) $3.25 - 0.25 \gtrless 3.75 - 0.25 \Rightarrow H_1$

iv) $6 - 3 \gtrless 3.75 - 0.25 \Rightarrow H_1$

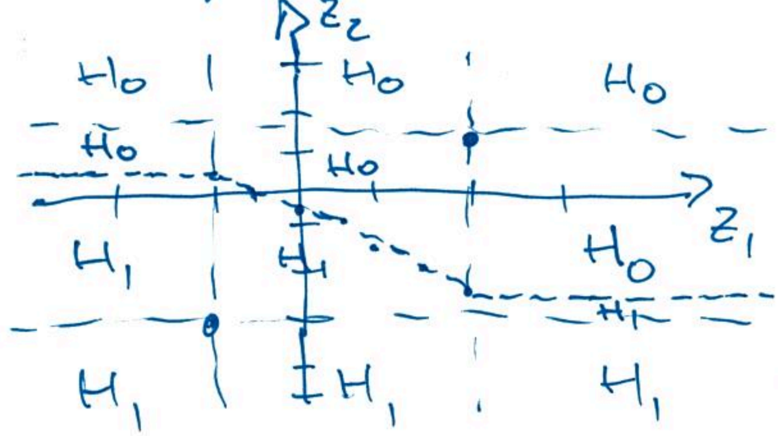
v) $6 - 3 \gtrless 4.25 - 0.25 \Rightarrow H_1$

g)

| $z_2$ \ $z_1$ | $z_1 < -1$ | $-1 \le z_1 < 2$ | $z_1 \ge 2$ |
|---|---|---|---|
| $z_2 < -3$ | $-(z_1+1)+(z_1-2) \underset{H_1}{\overset{H_0}{\gtrless}} -(z_2-1)+(z_2+3)$ $\Rightarrow -3 \gtrless 4 \Rightarrow \boxed{H_1}$ | $+(z_1+1)+(z_1-2) \underset{}{\overset{H_0}{\gtrless}} -(z_2-1)+(z_2+3)$ $\Rightarrow 2z_1 - 1 \gtrless 4$ $\Rightarrow z_1 \gtrless 5/2 \Rightarrow \boxed{H_1}$ | $(z_1+1)-(z_1-2) \overset{H_0}{\gtrless} 4$ $3 \gtrless 4 \Rightarrow \boxed{H_1}$ |
| $-3 \le z_2 < 1$ | $-3 \underset{H_1}{\overset{H_0}{\gtrless}} -2z_2 - 2$ $\Rightarrow z_2 \overset{H_0}{\gtrless} \frac{1}{2}$ | $2z_1 - 1 \underset{H_1}{\overset{H_0}{\gtrless}} -2z_2 - 2$ $z_1 + z_2 \underset{H_1}{\overset{H_0}{\gtrless}} -\frac{1}{2}$ | $+3 \underset{H_1}{\overset{H_0}{\gtrless}} -2z_2 - 2$ $\Rightarrow z_2 \underset{H_1}{\overset{H_0}{\gtrless}} -\frac{5}{2}$ $\boxed{H_0}$ |
| $z_2 \ge 1$ | $H_0$ | $H_0$ | $H_0$ |

| $z_2$ | $z_2 < -3$ | $-3 \le z_2 < 1$ | $z_2 \ge 1$ |
|---|---|---|---|
| $\lvert z_2 - 1\rvert - \lvert z_2 + 3\rvert$ | $4$ | $-2z_2 - 2$ | $-4$ |

| $z_1$ | $z_1 \le -1$ | $-1 \le z_1 < 2$ | $z_1 > 2$ |
|---|---|---|---|
| $\lvert z_1 + 1\rvert - \lvert z_1 - 2\rvert$ | $-3$ | $2z_1 - 1$ | $3$ |

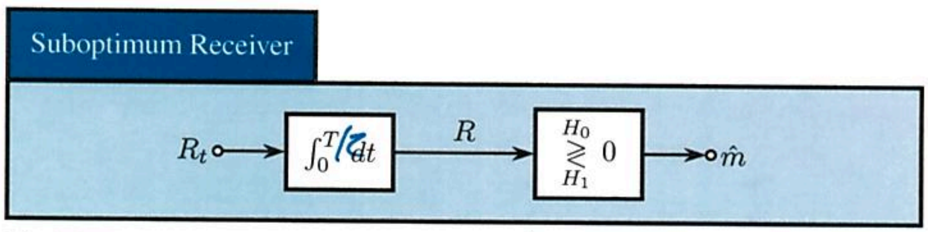


2. **Binary Signal Sets**
The following signal set is employed to transmit equally likely signals over an additive white Gaussian noise channel with spectral height $\frac{N_0}{2}$.

$$
\begin{aligned}
s_0(t) &= \sqrt{\tfrac{4E}{T}} \sin(2\pi \tfrac{t}{T}) \quad \text{for } 0 \le t \le \tfrac{T}{2} \\
s_1(t) &= -\sqrt{\tfrac{4E}{T}} \sin(2\pi \tfrac{t}{T}) \quad \text{for } 0 \le t \le \tfrac{T}{2}
\end{aligned}
$$

Note that both signals consist of exactly one *half* cycle of a sinusoidal signal.

(a) Sketch and accurately label the block diagram of a receiver that minimizes the probability of error.
(b) Compute the energy of each of the two signals.
(c) Compute the probability of error for your receiver from part (a).
(d) Consider now the following receiver:




Find the conditional distribution of the random variable $R$ at the output of the integrator for each of the two signals $s_0(t)$ and $s_1(t)$.
(e) Compute the probability of error achieved by the suboptimum receiver.
(f) Compare the probability of error for the suboptimum receiver to that of the optimum receiver. Express your answer in the form: "to achieve the same probability of error as the optimum receiver, the suboptimum system requires $a$ times more energy." Determine the factor $a$.
(g) Assume now that the received signal is corrupted by an interfering signal $x(t) = 10\cos(2\pi \frac{t}{T})$ so that the received signal under the $i$-th hypotesis $(i = 0, 1)$ is given by

$$H_i\colon R_t = s_i(t) + 10\cos(2\pi \frac{t}{T}) + N_t \quad \text{for } 0 \le t \le \frac{T}{2}$$

Compute the probability of error by the optimum receiver in the presence of the interfering signal.

(h) Explain your result in part (g).

**Hints:** The following relationships may be useful:

$$\begin{aligned}
\sin^2(x) &= \frac{1}{2}(1 - \cos(2x)) \\
\sin(x)\cos(x) &= \frac{1}{2}\sin(2x) \\
\int \cos(ax)dx &= \frac{1}{a}\sin(ax) \\
\int \sin(ax)dx &= -\frac{1}{a}\cos(ax)
\end{aligned}$$

a) $$R_t \longrightarrow \underset{\sin(2\pi t/T)}{\otimes} \longrightarrow \boxed{\int_0^{T/2}} \xrightarrow{R} \boxed{\gtrless 0} \longrightarrow \hat{m}$$

b) energy same for both signals:

$$E = \int_0^{T/2} \frac{4E}{T}\cdot \sin^2(2\pi t/T)\,dt$$

integrates to zero

$$= \frac{4E}{T}\cdot\frac{1}{2}\int_0^{T/2}(1-\cos(4\pi t/T))\,dt = \frac{4E}{T}\cdot\frac{1}{2}\cdot\frac{T}{2} = E$$

c) $$P_e = Q\left(\frac{\|s_0-s_1\|}{\sqrt{2N_0}}\right) \qquad \|s_0-s_1\|^2 = \|2\cdot s_0\|^2 \quad | s_1=-s_0$$
$$= 4\cdot\|s_0\|^2 = 4\cdot E$$
$$= Q\left(\frac{\sqrt{4E}}{\sqrt{2N_0}}\right) = Q\left(\sqrt{\frac{2E}{N_0}}\right)$$

d) $$H_0\ (s_0(t)):\ R \sim N\left(\int_0^{T/2} s_0(t)\,dt,\ \frac{N_0}{2}\cdot\frac{T}{2}\right)$$
$$H_1\ (s_1(t)):\ R \sim N\left(-\int_0^{T/2} s_0(t)\,dt,\ \frac{N_0}{2}\cdot\frac{T}{2}\right)$$
$$\int_0^{T/2} s_0(t)\,dt = \sqrt{\frac{4E}{T}}\int_0^{T/2}\sin(2\pi t/T)\,dt = -\sqrt{\frac{4E}{T}}\,\frac{\cos(2\pi t/T)}{2\pi/T}\Bigg|_0^{T/2}$$
$$= \frac{2\cdot T}{2\pi}\cdot\sqrt{\frac{4E}{T}} = 2\cdot\frac{\sqrt{E\cdot T}}{\pi}$$
$$\Rightarrow H_0:\ R \sim N\left(\frac{2\sqrt{TE}}{\pi},\ \frac{N_0}{2}\cdot\frac{T}{2}\right)$$
$$H_1:\ R \sim N\left(\frac{2\sqrt{TE}}{\pi},\ \frac{N_0}{2}\cdot\frac{T}{2}\right)$$

e) $$P_e = Q\left(\frac{2\sqrt{ET}/\pi}{\sqrt{N_0T}/2}\right) = \underline{Q\left(\frac{4}{\pi}\sqrt{\frac{E}{N_0}}\right)}$$

f) Energy must be increased by factor a s.t. ~~$(\pi/2)^2 \approx 2.5 \approx 4$dB~~
$a\cdot E\cdot\left(\frac{4}{\pi}\right)^2 \stackrel{!}{=} 2E$
$\Rightarrow a = \frac{\pi^2}{8} = 1.23 \approx 1$dB

g) opt. rec. frontend computes
$R = \langle R_t, s_0(t)\rangle$ but $\langle\cos(2\pi t/T), \sin(2\pi t/T)\rangle = \int_0^{T/2}\cos(2\pi t/T)\sin(2\pi t/T)\,dt = 0$
$\Rightarrow$ no impact on $P_e$.