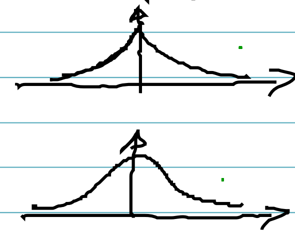
ECE 630 HW #8 - Solution Dr. Paris

1. H_0: $p_{R|H_0}(r) = \frac{1}{2} e^{-|r|}$

H_1: $p_{R|H_1}(r) = \frac{1}{\sqrt{2\pi}} e^{-r^2/2}$



π_0, π_1 $\qquad \frac{\pi_0}{\pi_1}$

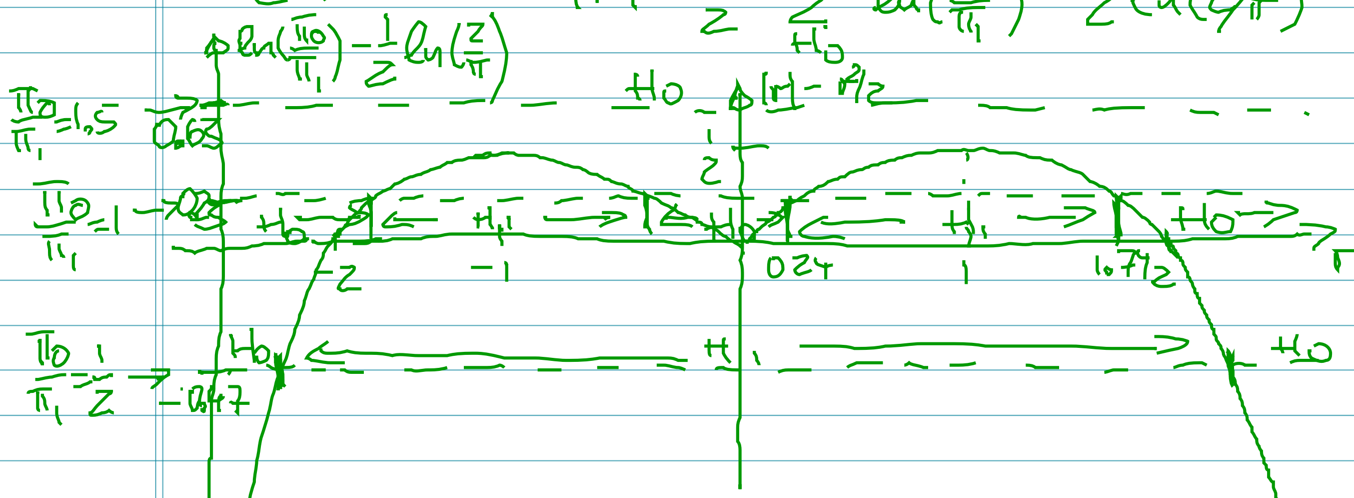
a) $$\Lambda(R) = \frac{p_{R|H_1}(r)}{p_{R|H_0}(r)} = \frac{\frac{1}{\sqrt{2\pi}} e^{-r^2/2}}{1/2\; e^{-|r|}}$$

$$= \sqrt{\frac{2}{\pi}} \exp\left(-\frac{r^2}{2} + |r|\right) \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\pi_0}{\pi_1}$$

b) log-likelihood:

$$L(r) = \frac{1}{2}\ln\left(\frac{2}{\pi}\right) + |r| - \frac{r^2}{2} \underset{H_0}{\overset{H_1}{\gtrless}} \ln\frac{\pi_0}{\pi_1}$$

$$\Leftrightarrow \quad |r| - \frac{r^2}{2} \underset{H_0}{\overset{H_1}{\gtrless}} \ln\left(\frac{\pi_0}{\pi_1}\right) - \frac{1}{2}\ln(2/\pi)$$



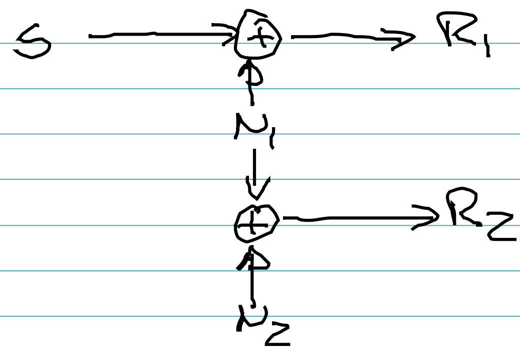
$\pi_0/\pi_1 = \frac{1}{2}$: decide H_1 for $|r| \le 2.39$, H_0 otherwis

$\pi_0/\pi_1 = 1$: decide H_1 for $0.24 \le |r| \le 1.74$, H_0 otherwise

$\pi_0/\pi_1 = 1.5$: decide H_0 always

if $\frac{\pi_0}{\pi_1} \ge \sqrt{\frac{2e}{\pi}} \Rightarrow$ always H_0

2. $\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = \begin{pmatrix} S \\ 0 \end{pmatrix} + \begin{pmatrix} N_1 \\ N_1 + N_2 \end{pmatrix}$



$\begin{pmatrix} R_1 \\ R_2 \end{pmatrix}$ are jointly Gaussian

$\Rightarrow$ mean and covariance

$$E\left[\begin{pmatrix} R_1 \\ R_2 \end{pmatrix}\right] = \begin{pmatrix} S \\ 0 \end{pmatrix}$$

$$K = \begin{pmatrix} E[(R_1 - S)^2] & E[(R_1 - S) R_2] \\ E[R_2 (R_1 - S)] & E[R_2^2] \end{pmatrix}$$

$$= \begin{pmatrix} E[N_1^2] & E[N_1 \cdot (N_1 + N_2)] \\ E[(N_1 + N_2) \cdot N_1] & E[(N_1 + N_2)^2] \end{pmatrix}$$

$$= \begin{pmatrix} \sigma^2 & \sigma^2 \\ \sigma^2 & 2\sigma^2 \end{pmatrix}$$

$$K^{-1} = \frac{1}{\sigma^2} \cdot \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$H_0: \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \sim N\left(\overbrace{\begin{pmatrix} -\sqrt{E} \\ 0 \end{pmatrix}}^{\underline{\mu}_0}, K \right)$$
$S = -\sqrt{E}$

$$H_1: \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \sim N\left(\underbrace{\begin{pmatrix} \sqrt{E} \\ 0 \end{pmatrix}}_{\underline{\mu}_1}, K \right)$$
$S = \sqrt{E}$

Likelihood ratio:

$$L(\underline{r}) = \frac{\exp\left(-\frac{1}{2}(\underline{r}-\underline{\mu}_1)^T \cdot K^{-1} \cdot (\underline{r}-\underline{\mu}_1)\right)}{\exp\left(-\frac{1}{2}(\underline{r}-\underline{\mu}_0)^T \cdot K^{-1} \cdot (\underline{r}-\underline{\mu}_0)\right)} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

log-likelihood:

$$\cancel{\tfrac{1}{2}} \cdot \left((\underline{r}-\underline{\mu}_0)^T K^{-1} (\underline{r}-\underline{\mu}_0) - (\underline{r}-\underline{\mu}_1)^T K^{-1} (\underline{r}-\underline{\mu}_1) \right) \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

$$\Leftrightarrow 2 \cdot \underline{r}^T \underbrace{K^{-1}(\underline{\mu}_1 - \underline{\mu}_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \underbrace{\underline{\mu}_1^T \cdot K^{-1} \underline{\mu}_1} - \underbrace{\underline{\mu}_0^T K^{-1} \underline{\mu}_0}$$

$$K^{-1} \cdot (\underline{\mu}_1 - \underline{\mu}_0) = \frac{\sqrt{E}}{6} \cdot \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

$$\underline{\mu}_1^T K^{-1} \underline{\mu}_1 = \frac{2 \cdot E}{6^2} = \underline{\mu}_0^T K^{-1} \underline{\mu}_0$$

$$\Rightarrow 2 \cdot \underline{r}^T \cdot \frac{\sqrt{E}}{6} \cdot \begin{pmatrix} 4 \\ -2 \end{pmatrix} \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

$$\Rightarrow \underline{r}^T \cdot \begin{pmatrix} 4 \\ -2 \end{pmatrix} \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

$$\Rightarrow r_1 \cdot 4 - r_2 \cdot 2 \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

$$\boxed{r_1 - \frac{1}{2} r_2 \underset{H_0}{\overset{H_1}{\gtrless}} 0}$$

$a = -\frac{1}{2}$

$\gamma = 0$

d) $Z = R_1 - \frac{1}{2} R_2 = \underbrace{\begin{bmatrix} 1 & -\frac{1}{2} \end{bmatrix}}_{\underline{\omega}^T} \cdot \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}$

$H_0: Z = R_1 - \frac{1}{2} R_2 \sim N\left(\underline{\omega}^T \cdot \begin{pmatrix} s_0 \\ 0 \end{pmatrix}, \underline{\omega}^T \cdot K \cdot \underline{\omega}\right)$

$H_1: Z \sim N\left(\underline{\omega}^T \begin{pmatrix} s_1 \\ 0 \end{pmatrix}, \underline{\omega}^T K \underline{\omega}\right)$

$H_0: Z \sim N\left(-\sqrt{E}, \frac{\sigma^2}{2}\right)$

$H_1: Z \sim N\left(+\sqrt{E}, \frac{\sigma^2}{2}\right).$

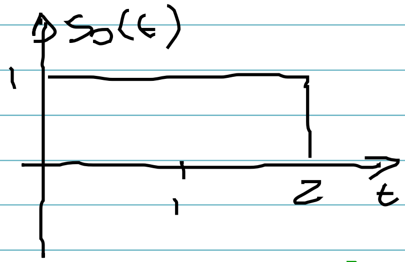
$\Rightarrow P_e = Q\left(\frac{\sqrt{E}}{\sqrt{\sigma^2/2}}\right) = Q\left(\sqrt{\frac{2E}{\sigma^2}}\right)$

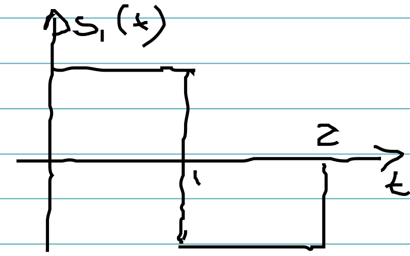
e) $H_0: R_1 \sim N(-\sqrt{E}, \sigma^2)$

$H_1: R_1 \sim N(+\sqrt{E}, \sigma^2).$

$\Rightarrow P_e = Q\left(\sqrt{\frac{E}{\sigma^2}}\right)$

3.





$E_0 = \|s_0(t)\|^2 = 2 \qquad E_1 = \|s_1(t)\|^2 = 2$

$\|s_0(t) - s_1(t)\|^2 = 4$

a) opt. receiver:

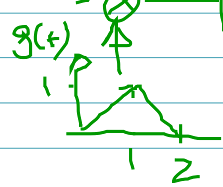
$$P_e = Q\left(\frac{\|s_0(t)-s_1(t)\|}{\sqrt{2N_0}}\right) = Q\left(\sqrt{\frac{2}{N_0}}\right)$$

b) $R_t \longrightarrow \boxed{\int_0^2 dt} \xrightarrow{R}$ $\quad H_0: R \sim N(2, N_0)$, $\quad H_1: R \sim N(0, N_0)$ $\Rightarrow P_e = Q\left(\sqrt{\frac{1}{N_0}}\right)$

c) $R_t \longrightarrow \boxed{\int_1^2 dt} \xrightarrow{R}$ $\quad$ equivalent to opt. receiver $\quad P_e = Q\left(\sqrt{\frac{2}{N_0}}\right)$

d) $R_t \longrightarrow \boxed{\int_0^1 dt} \xrightarrow{R}$ $\quad H_0: R \sim N(1, N_0/2)$, $\quad H_1: R \sim N(1, \frac{N_0}{2})$ $\Rightarrow P_e = \frac{1}{2}$

e) $R_t \longrightarrow \otimes \longrightarrow \boxed{\int_0^2 dt} \xrightarrow{R}$, with $g(t)$ into $\otimes$



$\langle s_0(t), g(t) \rangle = 1$

$\langle s_1(t), g(t) \rangle = 0$

$\|g(t)\| = \frac{2}{3}$

$H_0: R \sim N\left(1, \frac{N_0}{2} \cdot \frac{2}{3}\right)$

$H_1: R \sim N\left(0, \frac{N_0}{3}\right)$

$$\Rightarrow P_e = Q\left(\frac{1/2}{\sqrt{N_0/3}}\right) = Q\left(\sqrt{\frac{3}{4N_0}}\right)$$

4.

$$H_0: R_t = N_t \qquad 0 \le t \le T$$

$$H_1: R_t = \sqrt{\tfrac{E}{T}} + N_t \qquad 0 \le t \le T$$

a) $R_t \longrightarrow \bigotimes \longrightarrow \boxed{\int_0^T dt} \longrightarrow \boxed{\underset{H_0}{\overset{H_1}{\gtrless}} \frac{E}{2}} \longrightarrow$, with $\sqrt{E/T}$ into $\bigotimes$

b) $$P_e = Q\left(\frac{\| s_1(t) - s_0(t) \|}{\sqrt{2N_0}} \right) = Q\left(\sqrt{\frac{E}{2N_0}} \right)$$

c) $$\gamma(\lambda) = \lambda \cdot \|s_1\|^2 + (1-\lambda)\|s_0\|^2$$
$$= \lambda \cdot E$$

$$P_e = \frac{1}{2} P_{e|H_0}(\gamma) + \frac{1}{2} P_{e|H_1}(\gamma)$$

$$= \frac{1}{2} Q\left(\frac{\gamma}{\sqrt{\frac{N_0}{2} \cdot E}} \right) + \frac{1}{2} Q\left(\frac{E - \gamma}{\sqrt{\frac{N_0}{2} E}} \right)$$

$\gamma = \lambda E$: $$= \frac{1}{2} Q\left(\lambda \cdot \sqrt{\frac{2E}{N_0}} \right) + \frac{1}{2} Q\left((1-\lambda) \sqrt{\frac{2E}{N_0}} \right)$$