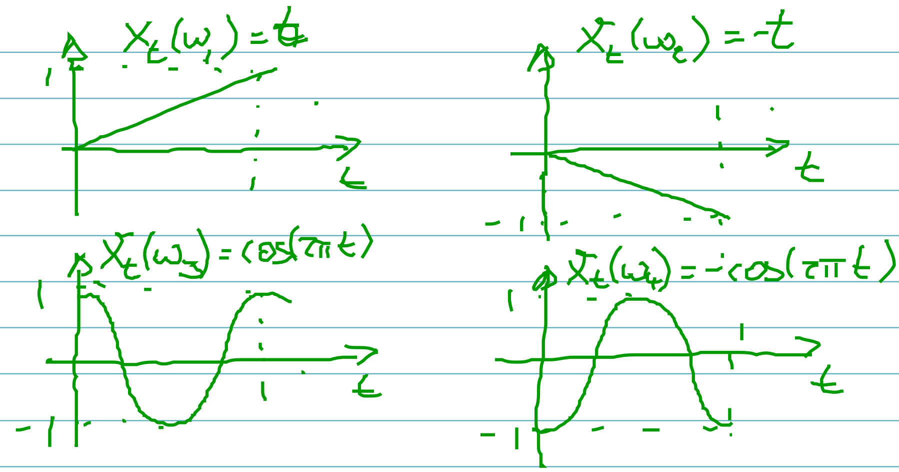
ECE 630 HW3 Solution Dr. Paris

1.



a) $Pr\{X_0(\omega)=1,\ X_1(\omega)=1\} = Pr\{\omega_3\} = \frac{1}{4}$

b) $Pr\{X_1(\omega)=1 \mid X_0(\omega)=0\} = \frac{Pr\{X_1(\omega)=1,\ X_0(\omega)=0\}}{Pr\{X_0(\omega)=0\}}$

$= \frac{Pr\{\omega_1\}}{Pr\{\omega_1, \omega_2\}}$

$= \frac{1/4}{1/2} = 1/2$

c) mean: $E[X_t(\omega)] = \frac{1}{4}\cdot\left[t + (-t) + \cos(2\pi t) + (-\cos(2\pi t))\right]$

$= 0$

$R_X(t,u) = \frac{1}{4}\cdot\left[t\cdot u + (-t)\cdot(-u) + \cos(2\pi t)\cdot\cos(2\pi u) + (-\cos(2\pi t))\cdot(-\cos(2\pi u))\right]$

$= \frac{1}{2}\, t\cdot u + \frac{1}{2}\cos(2\pi t)\cos(2\pi u)$

2.

a) $R_X(t,t) = E[X_t \cdot X_t] = E[X_t^2] \geq 0$ b/c $X_t^2 \geq 0$

b) $R_X(t,u) = E[X_t \cdot X_u]$
$= E[X_u \, X_t] = R_X(u,t)$

c) $E[(\underbrace{X_t \pm X_u}))^2] \geq 0$

$\Rightarrow E[X_t^2] \pm 2E[X_t \cdot X_u] + E[X_u^2] \geq 0$

$\Rightarrow R_X(t,t) \pm 2R_X(t,u) + R_X(u,u) \geq 0$

$\Rightarrow \dfrac{R_X(t,t) + R_X(u,u)}{2} \geq \mp R_X(t,u)$

$\Rightarrow \dfrac{R_X(t,t) + R_X(u,u)}{2} \geq |R_X(t,u)|$

For wss: $R_X(0) \geq |R_X(t)|$

d) $E[(X_t + aX_u)^2] \geq 0$

(*) $\Rightarrow R_X(t,t) + 2aR_X(t,u) + a^2 R_X(u,u) \geq 0$

$\min_a$: $\dfrac{d}{da}$: $2R_X(t,u) + 2aR_X(u,u) \overset{!}{=} 0$

$\Rightarrow a = -\dfrac{R_X(t,u)}{R_X(u,u)}$

$$R_X(t,t) - 2\cdot\underbrace{\left(\frac{R_X(t,u)}{R_X(u,u)}\right)}_{a}\cdot R_X(t,u) + \frac{R_X(t,u)^2}{R_X(u,u)^{\cancel{2}}}\cdot \cancel{R_X(u,u)} \geq 0$$

$$\Rightarrow R_X(t,t) - \frac{R_X(t,u)^2}{R_X(u,u)} \geq 0$$

$$\Rightarrow \boxed{R_X(t,t)\cdot R_X(u,u) \geq R_X(t,u)^2}$$

For wss: $\boxed{R_X(0)^2 \geq R_X(\tau)^2}$

3. $$X_t = \cos(2\pi F t)$$

$$F \sim U[0, f_0]$$

a) $$E[X_t] = E[\cos(2\pi F t)]$$

$$= \frac{1}{f_0}\int_0^{f_0} \cos(2\pi f t)\, df$$

$$= \frac{1}{f_0} \left.\frac{\sin(2\pi f t)}{2\pi t}\right|_0^{f_0} = \frac{\sin(2\pi f_0 t)}{2\pi f_0 t}$$

$$= \operatorname{sinc}(2\pi f_0 t)$$

$$R_X(t,u) = E[X_t \cdot X_u]$$

$$= E[\cos(2\pi F t) \cdot \cos(2\pi F u)]$$

$$= \frac{1}{f_0} \cdot \int_0^{f_0} \frac{1}{2}\Big(\cos(2\pi f(t-u)) + \cos(2\pi f(t+u))\Big)\, df$$

$$= \frac{1}{2}\operatorname{sinc}(2\pi f_0(t-u)) + \frac{1}{2}\operatorname{sinc}(2\pi f_0(t+u))$$

b) <u>not</u> WSS:
- $E[X_t]$ not constant
- $R_X(t,u)$ <u>not</u> $R_X(t-u)$

$F \sim U[0, f_0]$, $\Theta \sim U[-\pi, \pi)$

$X_t = \cos(2\pi F t + \Theta)$

c) $E[X_t] = E[\cos(2\pi F t + \Theta)]$

$= \frac{1}{f_0} \cdot \frac{1}{2\pi} \int_0^{f_0} \underbrace{\int_{-\pi}^{\pi} \cos(2\pi f t + \theta)\, d\theta}_{=0}\, df$

$= 0$

$R_X(t,u) = E[X_t \cdot X_u]$

$= \frac{1}{f_0} \frac{1}{2\pi} \int_0^{f_0} \int_{-\pi}^{\pi} \cos(2\pi f t + \theta) \cdot \cos(2\pi f u + \theta)\, d\theta\, df$

$= \frac{1}{f_0} \frac{1}{2\pi} \int_0^{f_0} \int_{-\pi}^{\pi} \frac{1}{2} \left(\cos(2\pi f (t-u)) + \cancel{\cos(2\pi f (t+u) + 2\theta)} \right) df\, d\theta$ $\to 0$

$= \frac{1}{f_0} \cdot \frac{1}{2} \int_0^{f_0} \cos(2\pi f (t-u))$

$= \frac{1}{2} \operatorname{sinc}(2\pi f_0 (t-u))$

d) $E[X_t]$ is const ; $R_X(t,u) = R_X(t-u) \Rightarrow$ WSS

e) First order pdf of X_t

$X_{t|F} = \cos(\underbrace{2\pi F t + \Theta}_{\text{mod}(2\pi F t + \Theta,\, 2\pi) \sim U[-\pi,\pi] = \Phi})$

$P_{X_t}(x)$ is pdf of $\cos(\Phi)$ $\Phi \sim U[-\pi, \pi)$

$$P_{X_{t|F}}(x) = \begin{cases} \frac{1}{\pi} \frac{1}{\sqrt{1-x^2}} & |x| < 1 \\ 0 & \text{else} \end{cases}$$

$\Rightarrow$ strict sense stationary (1^{st} order)