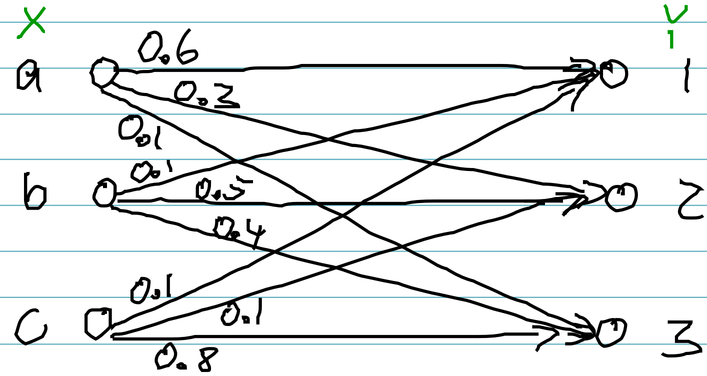
ECE 630 HW#1 - Solution Dr. Paris

1.



$Pr\{a\} = 0.3$

$Pr\{b\} = 0.5$

$Pr\{c\} = 0.2$

b) $Pr\{X,Y\} = Pr\{Y|X\} \cdot Pr\{X\}$ ←

X \ Y	1	2	3
a	0.18	0.09	0.03
b	0.05	0.25	0.2
c	0.02	0.02	0.16
P[Y]	0.25	0.36	0.39

a) $Pr\{X|Y\} = \dfrac{Pr\{X,Y\}}{Pr[Y]}$

X \ Y	1	2	3
a	0.72	0.25	0.08
b	0.2	0.69	0.51
c	0.08	0.06	0.41

Receiver / decoder:

$$Y=1 \rightarrow \hat{X}=a \leftarrow$$

$$Y=2 \rightarrow \hat{X}=b \leftarrow$$

$$Y=3 \rightarrow \hat{X}=c \leftarrow$$

$$P_{correct} = Pr\{\hat{X}=X\} \quad | \text{ sent} = \text{decision}$$

$$= \sum_{x=\{a,b,c\}} Pr\{\hat{X}=x \mid X=x\} \cdot Pr\{X=x\}$$

$$= Pr\{\hat{X}=a \mid X=a\} \cdot Pr\{X=a\} +$$
$$Pr\{\hat{X}=b \mid X=b\} \cdot Pr\{X=b\} +$$
$$Pr\{\hat{X}=c \mid X=c\} \cdot Pr\{X=c\}$$

$$= Pr\{Y=1 \mid X=a\} \cdot Pr\{X=a\} +$$
$$Pr\{Y=2 \mid X=b\} \cdot Pr\{X=b\} +$$
$$Pr\{Y=3 \mid X=c\} \cdot Pr\{X=c\}$$

$$= Pr\{Y=1, X=a\} + Pr\{Y=2, X=b\} +$$
$$Pr\{Y=3, X=c\}$$

$$= 0.18 + 0.25 + 0.16 = 0.59$$

$$\Rightarrow P_{error} = 1 - P_{correct}$$

$$= 0.41$$

d)

$X \backslash Y$	1	2	3
a	$\boxed{0.18}$	0.04	0.03
b	0.05	$\boxed{0.25}$	$\boxed{0.2}$
c	0.02	0.2	0.16

$Y=1 \qquad Y=2 \qquad Y=3$

$\downarrow \qquad \downarrow \qquad \downarrow$

$X=a \qquad X=b \qquad X=b$

e) $P_{correct} = P(Y=1, X=a) + P(Y=2, X=b) + P(Y=3, X=b)$

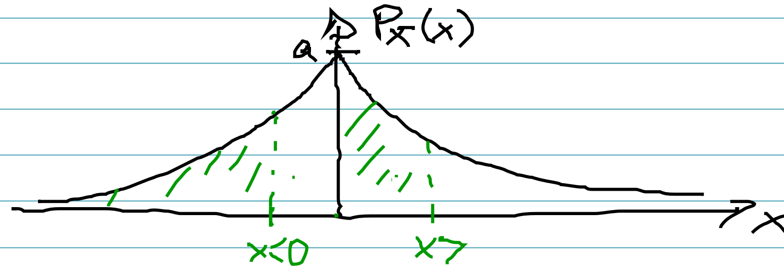
$= 0.18 + 0.25 + 0.2$

$= 0.63$

$\Rightarrow P_{error} = 1 - 0.63 = 0.37$

2.

$$P_X(x) = a \cdot e^{-b|x|} \qquad -\infty < x < \infty$$



a) i) $P_X(x) \geq 0 \quad \Rightarrow \quad a > 0$

ii) $\int_{-\infty}^{\infty} P_X(x) \overset{!}{=} 1$

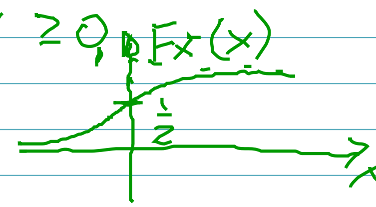
$$\int_{-\infty}^{\infty} a\, e^{-b|x|}\, dx = 2 \cdot \int_{0}^{\infty} a\, e^{-b \cdot x}\, dx$$

$$= \frac{2a}{b} \cdot \left(-e^{-bx}\right)\Big|_0^{\infty}$$

$$= \boxed{\frac{2a}{b} \overset{!}{=} 1} \leftarrow$$

b) $F_X(x) = \int_{-\infty}^{x} p_X(x)\, dx$

$$= \begin{cases} \int_{-\infty}^{x} a \cdot e^{+bz}\, dz & x < 0 \\ \frac{1}{2} + \int_{0}^{x} a \cdot e^{-bz}\, dz & x \geq 0 \end{cases}$$



$\frac{a}{b} = \frac{1}{2}$

$$= \begin{cases} \frac{a}{b}\, e^{bx} & x < 0 \\ \frac{1}{2} - \frac{a}{b}\, e^{-bz}\Big|_0^x & x > 0 \end{cases}$$

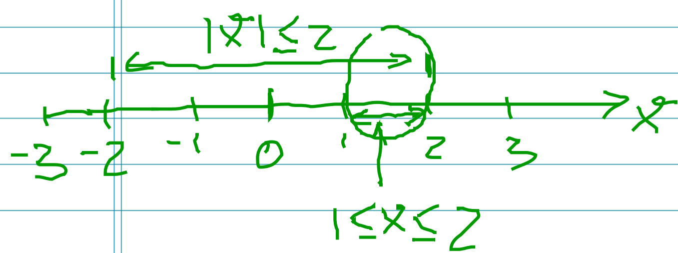
$$= \begin{cases} \frac{1}{2}\, e^{bx} & x < 0 \\ 1 - \frac{1}{2} e^{-bx} & x \geq 0 \end{cases}$$

c) $\Pr\{1 \le X \le 2\} = \Pr\{X \le 2\} - \Pr\{X \le 1\}$

$$= F_X(2) - F_X(1)$$

$$= \frac{1}{2}\left(e^{-b} - e^{-2b}\right)$$

d) $$\Pr\{1 \le X \le 2 \mid |X| \le 2\} = \frac{\Pr\{1 \le X \le 2 \text{ and } |X| \le 2\}}{\Pr\{|X| \le 2\}}$$



$$= \frac{\Pr\{1 \le X \le 2\}}{\Pr\{-2 \le X \le 2\}}$$

$$= \frac{F_X(2) - F_X(1)}{F_X(2) - F_X(-2)}$$

$$= \frac{1}{2}\,\frac{e^{-b} - e^{-2b}}{1 - e^{-2b}}$$

3. $Y_n = \max(X_1, X_2, \ldots, X_n)$ X_i iid

$$F_{Y_n}(y) = \Pr\{Y_n \le y\}$$
$$= \Pr\{\max(X_1, X_2, \ldots X_n) \le y\}$$
$$= \Pr\{X_1 \le y \text{ and } X_2 \le y \ldots \text{ and } \ldots X_n \le y\}$$
$$= \Pr\{X_1 \le y\} \cdot \Pr\{X_2 \le y\} \cdots \Pr\{X_n \le y\}$$
$$= F_{X_1}(y) \cdot F_{X_2}(y) \cdots F_{X_n}(y)$$
$$= (F_X(y))^N$$

$\Rightarrow$ pdf:

$$P_{Y_n}(y) = \frac{dF_{Y_n}(y)}{dy} = N \cdot \frac{dF_X(y)}{dy} \cdot (F_X(y))^{N-1}$$
$$= \underline{N \cdot P_X(y) \cdot (F_X(y))^{N-1}}$$

b) with $p_X(x) = \begin{cases} e^{-x} & \text{for } x \ge 0 \\ 0 & \text{for } x < 0 \end{cases}$

$$F_X(x) = \begin{cases} 1 - e^{-x} & \text{for } x \ge 0 \\ 0 & \text{else} \end{cases}$$

$$P_{Y_N}(y) = N \cdot e^{-y} \cdot (1 - e^{-y})^{N-1} \qquad y \ge 0$$

$N=1$: $P_{Y_1}(y) = e^{-y}$ $y \geq 0$

$$E[Y_1] = \int_0^{\infty} y \cdot e^{-y}\, dy$$

$$= -y \cdot e^{-y} \Big|_0^{\infty} + \int_0^{\infty} e^{-y}\, dy$$

$$= -y \cdot e^{-y} - e^{-y} \Big|_0^{\infty} = \underline{1}$$

$N=2$ $P_{Y_2}(y) = 2 \cdot e^{-y} \cdot (1-e^{-y})$

$$E[Y_2] = \int_0^{\infty} y \cdot 2e^{-y}(1-e^{-y})\, dy = \frac{3}{2}$$

Note:
- 'm' may denote 'meter' or 'milli'
- spelled out when ambiguous

4. Path Loss and SNR

a) Friis equation:

(linear): $$L_p = \left(\frac{c}{4\pi f_c d}\right)^2$$

(dB): $$10\cdot\log_{10}(L_p) = 20\cdot\log_{10}\left(\frac{c}{4\pi}\right) - 20\log_{10}(f_c) - 20\log_{10}(d)$$

$$= 147\,\text{dB}\,\frac{\text{meter}}{s} - 20\cdot\log_{10}\left(\frac{f_c}{Hz}\right) - 20\log_{10}\left(\frac{d}{m}\right)$$

b) $$L_{P|dB} = 147\,\text{dB}\,\frac{\text{meter}}{s} - \underbrace{180\,\text{dBHz}}_{=1\,GHz} - 20\cdot\log_{10}\left(\frac{d}{m}\right)$$

$$= -33\,\text{dBmeter} - 20\log_{10}\left(\frac{d}{m}\right)$$

$$\Rightarrow P_{r|dBm} = P_{t|dBm} + L_{P|dB}$$

$$= 10\,\text{dBm} - 33\,\text{dBmeter} - 20\cdot\log_{10}\left(\frac{d}{m}\right)$$

$$= -23\,\text{dBmW}\cdot\text{meter} - 20\cdot\log_{10}\left(\frac{d}{m}\right)$$

c) $$SNR = \frac{P_r}{P_N} \Rightarrow SNR_{dB} = P_{r|dBm} - P_{N|dBm}$$

$$P_N = K\cdot T\cdot W \Rightarrow \underset{(@T=300K)}{P_{N|dBm}} = -174\,\frac{\text{dBm}}{Hz} + \underbrace{10\cdot\log_{10}\left(\frac{W}{Hz}\right)}_{=70\,\text{dBmHz}}$$

$$= -104\,\text{dBm}$$

$$\Rightarrow SNR_{dB} = -23\,\text{dBmW}\cdot\text{meter} - 20\cdot\log_{10}\left(\frac{d}{m}\right) - (-104\,\text{dBm})$$

$$= 81\,\text{dBmeter} - 20\log_{10}\left(\frac{d}{m}\right) \overset{!}{\geq} 10\,\text{dB}$$

$$\Rightarrow 20\log_{10}\left(\frac{d}{m}\right) \leq 71\,\text{dBmeter}$$

$$\Rightarrow d \leq 10^{71/20}\,m = 3548\,m$$