ECE 630 HW11 - Solution Dr. Pari

1) P.3.11

a) With respect to the given basis, the representations are:

Signal set A:

$$\underline{s}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \underline{s}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \underline{s}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \underline{s}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Signal set B:

$$\underline{s}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \underline{s}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \underline{s}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \underline{s}_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

b) Signal set A:

- all signals are separated by $d = d_{min} = \sqrt{2}$
- $E_s = 1 \Rightarrow E_b = \frac{E_s}{\log_2 4} = \frac{1}{2}$
- 3 nearest neighbors

$\Rightarrow \eta = \frac{d_{min}^2}{E_b} = 4$

$\Rightarrow P_e(A) \leq 3 \cdot Q\left(\sqrt{\frac{\eta \cdot E_b}{2N_0}}\right) = 3Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$

Signal set B:

- each signal has
  - 2 neighbors at $d = d_{min} = \sqrt{2}$
  - 1 neighbor at $d = 2 = \sqrt{2} \cdot d_{min}$
- $E_s = 2 \Rightarrow E_b = \frac{2}{2} = 1 \Rightarrow \eta = \frac{d_{min}^2}{E_b} = 2$

$$\Rightarrow P_e \le 2\cdot Q\left(\sqrt{\frac{\eta E_b}{2N_0}}\right) + 1\cdot Q\left(\sqrt{\frac{2\eta E_b}{2N_0}}\right) = 2Q\left(\sqrt{\frac{E_b}{N_0}}\right) + Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

At high SNR, the penalty for using signal set B is given by the ratio of the efficiencies:

$$\frac{\eta_A}{\eta_B} = \frac{4}{2} = 2$$ ⇒ signal set B requires double E_b to achieve same P_e at high SNR.

c) It is possible to find exact P_e for set B because set B actually forms a 2-dimensional square. To see that, perform two transformations that preserve geometry, i.e. don't change P_e.

1.) Translation – subtract $\underline{m} = -0.5 \cdot \begin{pmatrix}1\\1\\1\\1\end{pmatrix}$

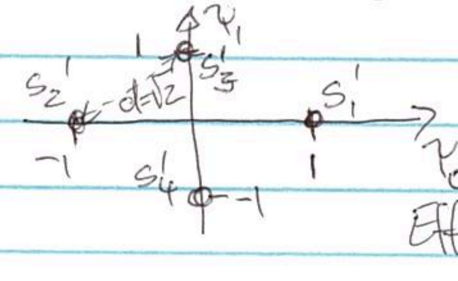
$$s_1' = \begin{pmatrix}0.5\\-0.5\\-0.5\\0.5\end{pmatrix} \quad \underline{s}_2' = \begin{pmatrix}-0.5\\0.5\\0.5\\-0.5\end{pmatrix} \quad \underline{s}_3' = \begin{pmatrix}0.5\\-0.5\\0.5\\-0.5\end{pmatrix} \quad \underline{s}_4' = \begin{pmatrix}-0.5\\0.5\\-0.5\\0.5\end{pmatrix}$$

- Note that $s_1' = -s_2'$ and $s_3' = s_4'$ and $s_1' \perp s_3'$
- Also, $\|s_n'\| = 1$ for $n = 1,2,3,4$

2.) Rotation / change of orthonormal basis

$\psi_0 = s_1' \qquad \psi_1 = s_3'$

Then:



Signals form a square with side $d = d_{min} = \sqrt{2}$

Same geometry as QPSK:

$\Rightarrow P_e = 2Q\left(\sqrt{\frac{E_b}{N_0}}\right) - Q^2\left(\sqrt{\frac{E_b}{N_0}}\right)$

Efficiency is still $\eta_B = 2$!!

2) P. 3.18

a) An error occurs, when the noise in either of the two dimensions exceeds

$$N > \frac{d}{2} + \frac{d_1}{2} = (1+\alpha)\frac{d}{2}$$

Also, $E_s = 2 \cdot \left(\frac{d}{2}\right)^2 = \frac{d^2}{2} \Rightarrow E_b = \frac{d^2}{4} \Rightarrow \eta = 4$

$$p = P_e \leq 2 \cdot Q\left(\frac{(1+\alpha)d}{2 \cdot \sqrt{N_0/2}}\right) = 2Q\left(\sqrt{\frac{(1+\alpha)^2 \cdot 2E_b}{N_0}}\right)$$

$$= 2Q\left((1+\alpha)\sqrt{\frac{2E_b}{N_0}}\right)$$

Similarly, an erasure occurs if the noise in either of the two dimensions is

$$\frac{d}{2} - \frac{d_1}{2} \leq N \leq \frac{d}{2} + \frac{d_1}{2}$$

$$\text{or} \quad (1-\alpha)\frac{d}{2} \leq N \leq (1+\alpha)\frac{d}{2}$$

$$\Rightarrow q \leq 2 \cdot \left[Q\left((1-\alpha)\sqrt{\frac{2E_b}{N_0}}\right) - Q\left((1+\alpha)\sqrt{\frac{2E_b}{N_0}}\right)\right]$$

b) An error ocurs if one of the disjoint events occurs

- $N_I > (1+\alpha)\frac{d}{2}$ <u>and</u> $N_Q < (1-\alpha)\frac{d}{2}$
- $N_I < (1-\alpha)\frac{d}{2}$ <u>and</u> $N_Q > (1+\alpha)\frac{d}{2}$
- $N_I > (1+\alpha)\frac{d}{2}$ <u>and</u> $N_Q > (1+\alpha)\frac{d}{2}$

$$\Rightarrow p = P_e = 2 \cdot Q\left((1+\alpha)\sqrt{\frac{2E_b}{N_0}}\right) \cdot \left(1 - Q\left((1-\alpha)\sqrt{\frac{2E_b}{N_0}}\right) + Q^2\left((1+\alpha)\sqrt{\frac{2E_b}{N_0}}\right)\right)$$

For erasures, we must correct the union bound above by subtracting the probability that both noise components fall into the erasure region:

Let q_1 be the probability that one of the noise components falls into the erasure region:

$$q_1 = \Pr\left\{(1-\alpha)\frac{d}{2} < N < (1+\alpha)\frac{d}{2}\right\}$$
$$= Q\left((1-\alpha)\sqrt{\frac{2E_b}{N_0}}\right) - Q\left((1+\alpha)\sqrt{\frac{2E_b}{N_0}}\right)$$

Then, $q = 2q_1 - q_1^2$

$$= 2\cdot\left(Q\left((1-\alpha)\sqrt{\frac{2E_b}{N_0}}\right) - Q\left((1+\alpha)\sqrt{\frac{2E_b}{N_0}}\right)\right) -$$
$$\left(Q\left((1-\alpha)\sqrt{\frac{2E}{N_0}}\right) - Q\left((1+\alpha)\sqrt{\frac{2E_b}{N_0}}\right)\right)^2$$

c) Numerical solution:

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% P3_18_c.m - Madhow Problem 3.18 c)

EbN0dB = 4;
EbN0 = 10^(EbN0dB/10);

alpha = 0:0.0001:0.2;

p = 2*Q((1+alpha)*sqrt(2*EbN0));
q = 2*Q((1-alpha)*sqrt(2*EbN0)) - p;

[~,ind] = min(abs(q-2*p));
alpha_ind = alpha(ind);
val_ind = q(ind);

plot(alpha, 2*p, alpha, q, alpha_ind, val_ind, 'o');
grid on
xlabel('\alpha = d_1/d')
legend('2p','q',sprintf('\alpha = %5.3f', alpha_ind));
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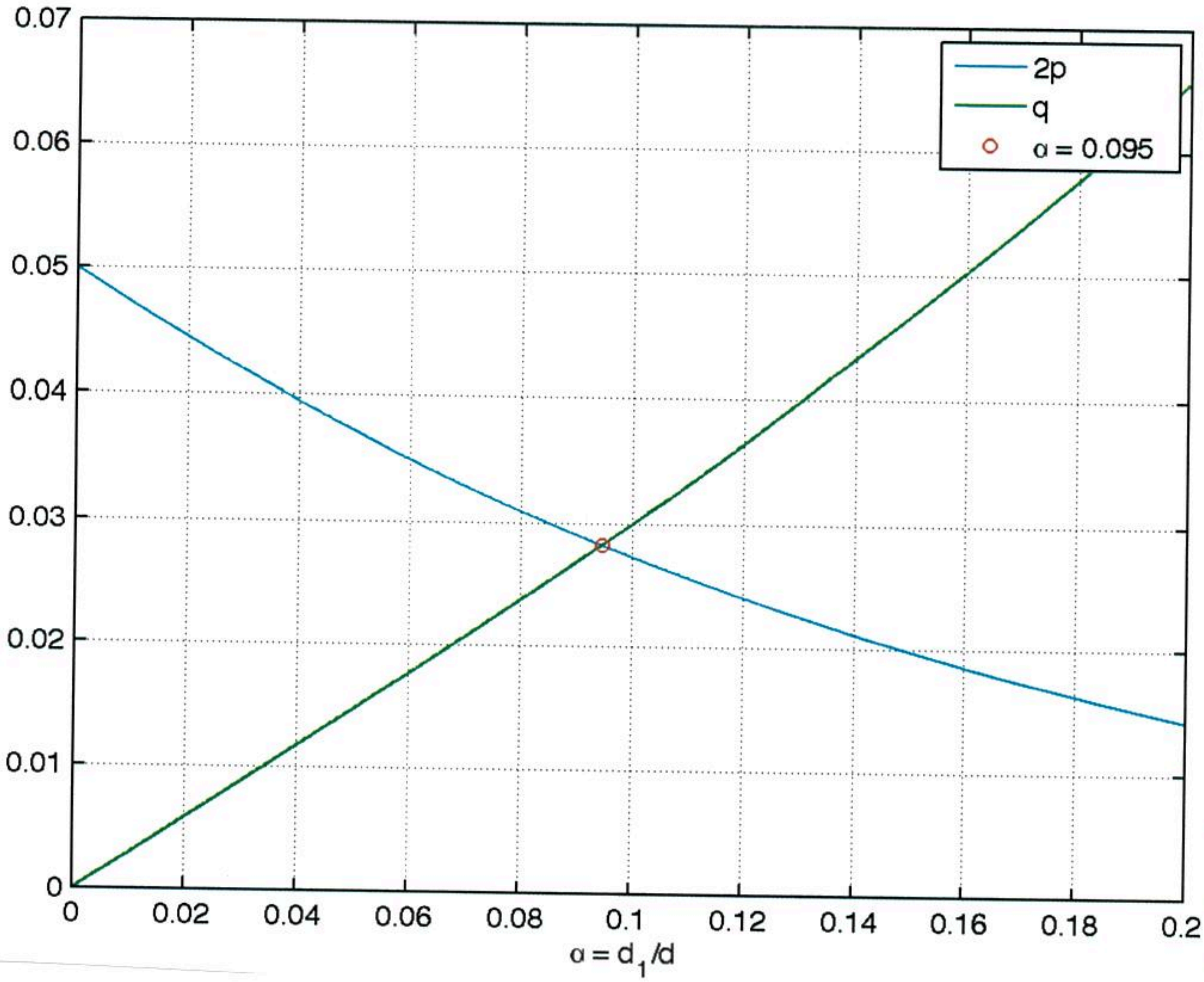
$\alpha \approx 0.095$ yields $q = 2p$.

See plot on next page



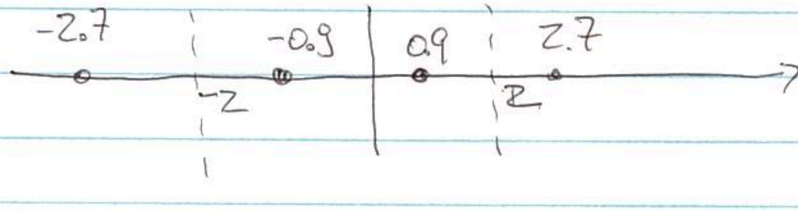
3) P. 3.20

a) For the two inner points:

$$P_e = Q\left(\frac{0.9}{\sqrt{N_0/2}}\right) + Q\left(\frac{1.1}{\sqrt{N_0/2}}\right)$$

For the outer points:

$$P_e = Q\left(\frac{0.7}{\sqrt{N_0/2}}\right)$$



Symbol energy: $E_s = \frac{1}{2}(0.9^2 + 2.7^2) = 5 \cdot 0.9^2$

$$\Rightarrow E_b = \frac{1}{2} E_s = \frac{5}{2} 0.9^2$$

Expressions of the form $Q\left(\frac{d}{\sqrt{N_0/2}}\right)$ become:

$$Q\left(\frac{d}{\sqrt{N_0/2}}\right) = Q\left(\frac{d}{\sqrt{E_b}} \cdot \sqrt{\frac{2E_b}{N_0}}\right) = Q\left(\sqrt{\frac{d^2}{E_b}} \cdot \sqrt{\frac{2E_b}{N_0}}\right)$$

$$\Rightarrow \text{inner points: } P_e = Q\left(\sqrt{\frac{2}{5}}\sqrt{\frac{2E_b}{N_0}}\right) + Q\left(\frac{11}{9}\sqrt{\frac{2}{5}}\sqrt{\frac{2E_b}{N_0}}\right)$$

$$= Q\left(\sqrt{\frac{4E_b}{5N_0}}\right) + Q\left(\frac{11}{9}\sqrt{\frac{E_b \cdot 4}{5N_0}}\right)$$

$$\text{outer points: } P_e = Q\left(\frac{7}{9}\sqrt{\frac{4E_b}{5N_0}}\right)$$

$$\Rightarrow \text{Avg. } P_e = \frac{1}{2} \cdot \left[Q\left(\frac{7}{9}\sqrt{\frac{4E_b}{5N_0}}\right) + Q\left(\sqrt{\frac{4E_b}{5N_0}}\right) + Q\left(\frac{11}{9}\sqrt{\frac{4E_b}{5N_0}}\right)\right]$$

At high SNR, mismatch causes degradation of $20 \cdot \log_{10}(7/9) = -2.2\,dB$

b) Proceeding similarly, we have relevant distances: $d = 0.9$, $d = 1.1$, and $d = 1.3$

Symbol energy: $E_s = 5 \cdot 1.1^2 \Rightarrow E_b = \frac{5}{2} 1.1^2$

$$\Rightarrow \text{Avg. } P_e = \frac{1}{2}\left[Q\left(\frac{9}{11}\sqrt{\frac{4E_b}{5N_0}}\right) + Q\left(\sqrt{\frac{4E_b}{5N_0}}\right) + Q\left(\frac{13}{11}\sqrt{\frac{4E_b}{5N_0}}\right)\right]$$

At high SNR, mismatch causes degradation of $20 \cdot \log_{10}\left(\frac{9}{11}\right) = -1.74\,dB$

c) In (a), the AGC circuit overestimates the power and scales the signal too aggressively. Overestimation occurs for example because the observed power is signal power plus noise power.

4.) P6.1

Given: Rate: $R_s = 1$ Gbit/s

BW: $B = 1.5$ GHz

excess BW: $\alpha = 50\%$

$\Rightarrow$ eff. BW: $B_{min} = \frac{B}{1+\alpha} = 1$ GHz

noise figure: $F = 6$ dB

Shannon - Capacity:

$$R_s \leq C = B_{min} \cdot \log_2\left(1 + \frac{P_{RX}}{N_0 \cdot B_{min}}\right)$$

Noisepower density:

$$N_0 = KT_0 \cdot 10^{F/10} = (1.38 \cdot 10^{-23}) \cdot (290) \cdot 4 \frac{W}{Hz}$$

$$= 1.6 \cdot 10^{-20} \frac{W}{Hz} = 1.6 \, 10^{-17} \frac{mW}{Hz}$$

$$= -168 \frac{dBm}{Hz}$$

Noise Power:

$$N_0 \cdot B_{min} = -168 \frac{dBm}{Hz} + \underbrace{90\,dBHz}_{10\log_{10}(1GHz)} = \underline{\underline{-78\,dBm}}$$

Solve Capacity for P_{RX}!

$$P_{RX_{dBm}} = N_0 B_{min} \cdot \underbrace{\left(2^{R_s/B_{min}} - 1\right)}_{=1} = N_0 B_{min} = -78\,dBm$$

Accounting for antenna gains and path loss:

$$P_{RX,dBm} = P_{TX,dBm} + \underbrace{2 \cdot G_{T,dB}}_{\text{antenna gain}} + \underbrace{20 \cdot \log_{10}\left(\frac{\lambda}{4\pi r}\right)}_{\text{path loss}}$$

$G_{T,dB} = 10\,dB$

Given: $r = 1000\,m$

assume: $f_c = 10\,GHz$ (not given)

$$\Rightarrow \lambda = \frac{c}{f_c} = \frac{3 \cdot 10^8}{10^{10}} = 3 \cdot 10^{-2}\,m$$

$$\Rightarrow \text{Pathloss} = -112.5\,dB$$

$$\Rightarrow P_{TX,dBm} = P_{RX,dBm} - 20\,dB + 112.5\,dB = 14.5\,dBm$$

$$= \underline{\underline{27\,mW}}$$