ECE 630 HW #10 - Solution Dr. Paris

1. P. 3.12

a) 8-PSK: $E_s = R^2 \Rightarrow E_b = \frac{R^2}{3}$

QAM 1: d_1 = distance between nearest neigbors

$$E_s = \frac{4}{8}\cdot \underbrace{2\left(\frac{d_1}{2}\right)^2}_{\text{4 inner pts}} + \underbrace{\frac{4}{8}\cdot\left(\left(\frac{d_1}{2}\right)^2+\left(\frac{3d_1}{2}\right)^2\right)}_{\text{4 outer pts}}$$

$$= \frac{3}{2}d_1^2 \Rightarrow E_b = \frac{d_1^2}{2}$$

QAM2: d_2 = spacing between nearest neigbors

$$E_s = \frac{2}{8}\cdot\left(\frac{d_2}{2}\right)^2 + \frac{4}{8}\cdot\left(\left(\frac{d_2}{2}\right)^2+d_2^2\right)+\frac{2}{8}\cdot\left(\frac{3d_2}{2}\right)^2$$

$$= \frac{5}{4}d_2^2 \Rightarrow E_b = \frac{5}{12}d_2^2$$

Want all three constellations to have same E_b:

8PSK v. QAM1: $\frac{R^2}{3} \doteq \frac{d_1^2}{2} \Rightarrow R^2 = \frac{3}{2}d_1^2 \Rightarrow R = \sqrt{\frac{3}{2}}\,d_1$

QAM2 v. QAM1: $\frac{5}{12}d_2^2 \doteq \frac{d_1^2}{2} \Rightarrow d_2^2 = \frac{6}{5}d_1^2 \Rightarrow d_2 = \sqrt{\frac{6}{5}}\,d_1$

b) Compute efficiency $\eta = \frac{d_{min}^2}{E_b}$ for each constellation

8PSK: $d_{min} = R\cdot\sqrt{2-\sqrt{2}} \Rightarrow d_{min}^2 = R^2\cdot(2-\sqrt{2})$

$E_b = \frac{R^2}{3} \Rightarrow \eta = \frac{2-\sqrt{2}}{1/3} = 6-3\sqrt{2} \approx 1.76$

QAM1: $d_{min} = d_1 \Rightarrow d_{min}^2 = d_1^2$

$E_b = \frac{d_1^2}{2} \Rightarrow \eta = \frac{1}{1/2} = 2$

QAM2: $d_{min} = d_2 \Rightarrow d_{min}^2 = d_2^2$

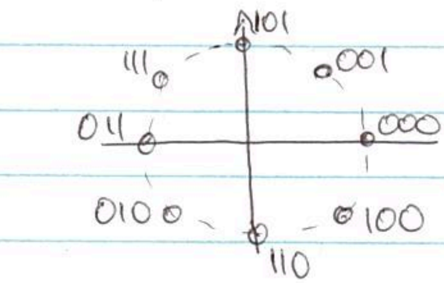
$E_b = \frac{5}{12} d_2^2 \Rightarrow \eta = \frac{1}{5/12} = \frac{12}{5} = 2.4$

Therefore, at high SNR expect:

$$P_e(\text{QAM2}) < P_e(\text{QAM1}) < P_e(\text{8PSK})$$

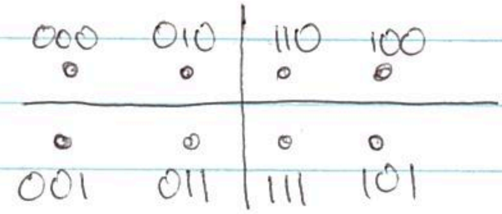
c) 8-PSK: Gray code exists! (2 nearest neighbors)

E.g.:



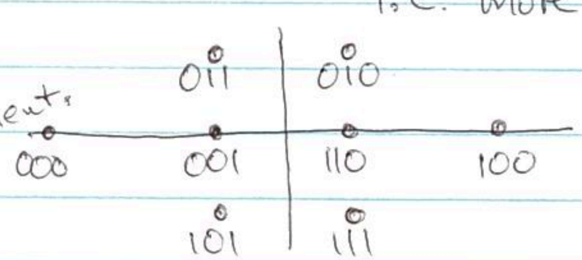
QAM1: Gray code exists! (at most 3 nearest neighbors)

Eg.



QAM2: No Gray code – some nodes have 4 neighbors i.e. more than # of bits.

Possible bit assignment:



Outer pts are Gray coded.

d) For the two Gray-coded constellations, we rely on

$$\Pr[\text{bit error}] \approx \frac{1}{3}\Pr[\text{symbol error}]$$

and

$$\Pr[\text{symbol error}] \approx \bar{N}_{d_{min}} \cdot Q\left(\sqrt{\eta \cdot \frac{E_b}{2N_0}}\right)$$ nearest neigbor approximation

8-PSK: $\Pr[\text{bit error}] \approx \frac{2}{3} \cdot Q\left(\sqrt{\frac{3(2-\sqrt{2})E_b}{2N_0}}\right)$

$\bar{N}_{d_{min}} = 2$

QAM 1: $\bar{N}_{d_{min}} = \frac{5}{2}$

$$\Pr[\text{bit error}] \approx \frac{5}{6} Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

For QAM 2, the bit error probability depends on the specific coding. For the coding above, the number of bit differences with all nearest neighbors differs for each symbol b:

b	000	001	010	011	100	110	111	Avg.
# of neighbors	1	4	2	2	1	4	2	2
# of bit diffs	1	6	2	2	1	6	2	11/4

The average # of bit differences is $\frac{11}{4} = \bar{N}_{bit}$

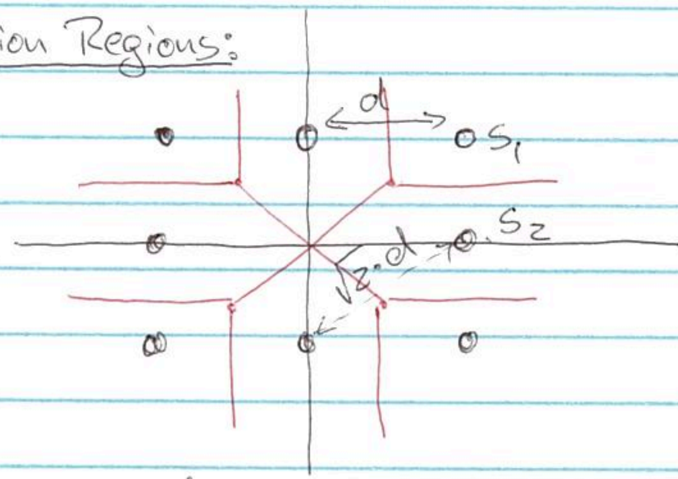
The probability of a bit error is then approximately:

$$\Pr[\text{bit error}] \approx \frac{\bar{N}_{bit}}{3} \cdot Q\left(\sqrt{\eta \frac{E_b}{2N_0}}\right)$$

$$= \frac{11}{12} \cdot Q\left(\sqrt{\frac{6E_b}{5N_0}}\right)$$

2) P.13.3:

a) Decision Regions:



b) Union bound is computed for the two typical points labeled s_1 and s_2 above

We begin by computing $\eta = \frac{d_{min}^2}{E_b}$

$$d_{min}^2 = d^2 \quad (\text{see above for } d)$$

$$E_s = \frac{4}{8} d^2 + \frac{4}{8} \cdot 2d^2 = \frac{3}{2} d^2 \Rightarrow E_b = \frac{d^2}{2}$$

$$\Rightarrow \eta = \frac{1}{1/2} = 2$$

Point s_1 shares decision boundaries with 2 nearest neighbors at distance d:

$$P_{e|s_1} \leq 2 \cdot Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right) = 2 \cdot Q\left(\sqrt{\frac{d_{min}^2}{E_b}} \cdot \sqrt{\frac{E_b}{2N_0}}\right)$$

$$= 2 \cdot Q\left(\sqrt{\eta \cdot \frac{E_b}{2N_0}}\right) = 2Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

Point s_2 shares decision boundaries with:

2 points at distance d

2 points at distance $\sqrt{2}\,d$

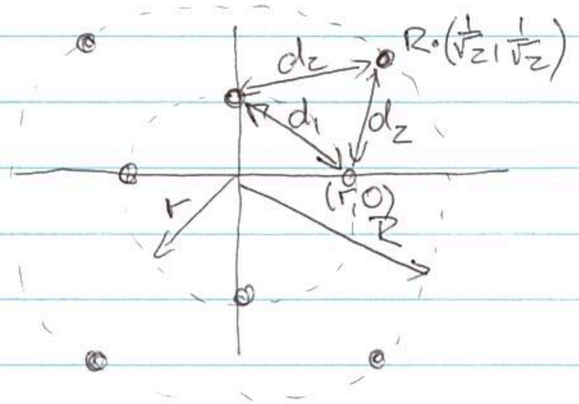
$$\Rightarrow P_{e|s_2} \leq 2 \cdot Q\left(\frac{d}{\sqrt{2N_0}}\right) + 2Q\left(\frac{\sqrt{2}\,d}{\sqrt{2N_0}}\right)$$

$$= 2 \cdot Q\left(\sqrt{E_b}\right) + 2Q\left(\sqrt{2E_b}\right)$$

$$P_e \le \frac{4}{8} P_{e|s_1} + \frac{4}{8} P_{e|s_2}$$

$$\le 2Q\left(\sqrt{\frac{E_b}{N_0}}\right) + Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

c) It is fairly obvious that the best efficiency is achieved when $d_1 = d_2$ in the following diagram:



$$d_1^2 = 2r^2 \qquad (1)$$

$$d_2^2 = \frac{R^2}{2} + \left(\frac{R}{\sqrt{2}} - r\right)^2$$

$$= R^2 + r^2 - \sqrt{2}\,R\cdot r \qquad (2)$$

For $d_1^2 = d_2^2$: $\frac{R}{r} = \frac{\sqrt{2}+\sqrt{6}}{2} \approx 1.93$

Symbol energy: $E_s = \frac{1}{2}r^2 + \frac{1}{2}R^2$

$$= \frac{1}{2}r^2 + \frac{1}{2}\left(\frac{\sqrt{2}+\sqrt{6}}{2}r\right)^2$$

$$= r^2 \cdot \frac{3+\sqrt{3}}{2}$$

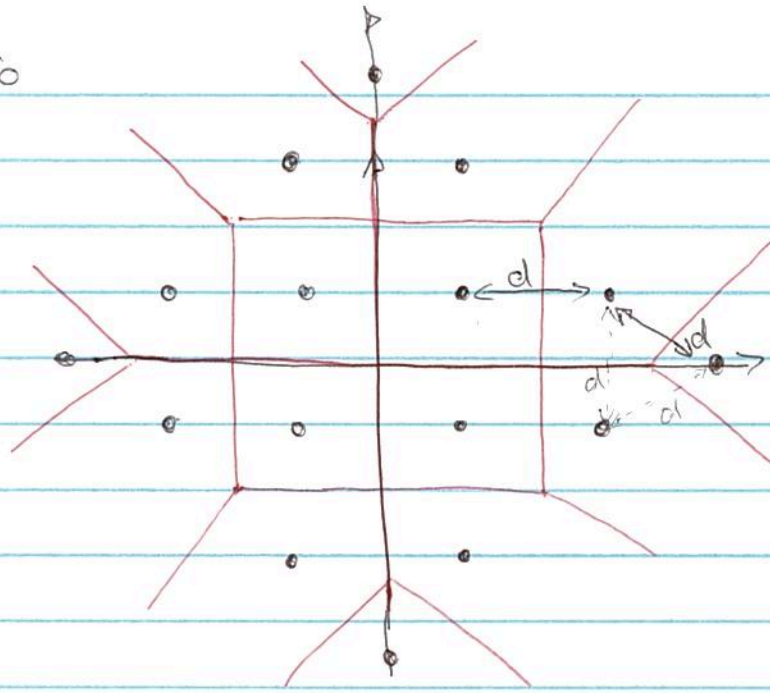
$$\Rightarrow E_b = r^2 \cdot \frac{3+\sqrt{3}}{6}$$

$$\Rightarrow \eta = \frac{d_{min}^2}{E_b} = \frac{2\cancel{r^2}}{\cancel{r^2}\,\frac{3+\sqrt{3}}{6}} = \frac{12}{3+\sqrt{3}} \approx 2.54$$

Better than $\eta = 2$ in part b.

3. P. 3.16

a)



b) $d_{min} = d$ for both regular 16-QAM and the above constellation.

For 16-QAM: $E_s = \frac{4}{16} 2\left(\frac{d}{2}\right)^2 + \frac{8}{16} \cdot \left(\left(\frac{d}{2}\right)^2 + \left(\frac{3d}{2}\right)^2\right) + \frac{4}{16} 2\left(\frac{3d}{2}\right)^2$

$= \frac{5}{2} d^2 \Rightarrow E_b = \frac{5}{6} d^2 \Rightarrow \eta = \frac{6}{5} = 1.2$

For above: $E_s = \frac{4}{16} 2\left(\frac{d}{2}\right)^2 + \frac{8}{16}\left(\left(\frac{d}{2}\right)^2 + \left(\frac{3d}{2}\right)^2\right) + \frac{4}{16} \cdot \left(\frac{3}{2}d + \frac{\sqrt{3}}{2}d\right)^2$

moved points are at distance: $\frac{3}{2}d + \frac{\sqrt{3}}{2}d$

$= \frac{17 + 3\sqrt{3}}{8} \cdot d^2$

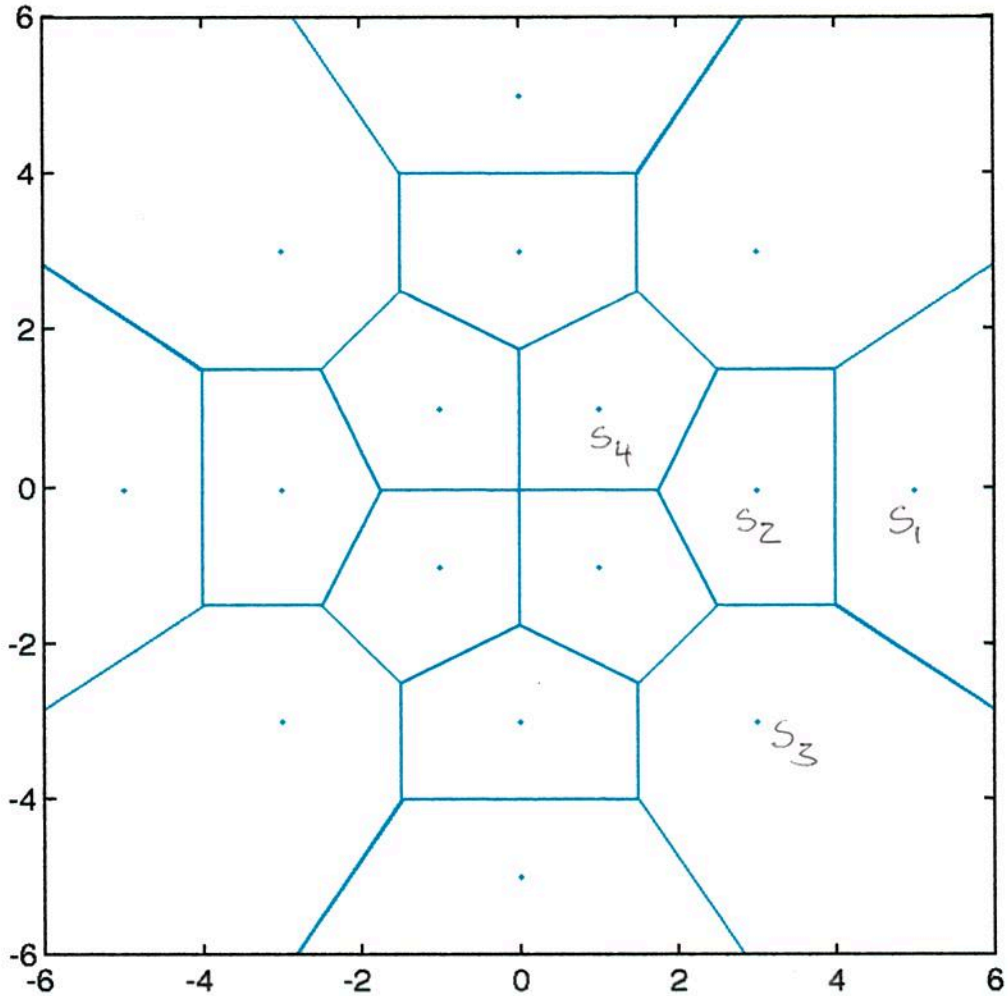
$\approx 2.77\, d^2 \Rightarrow E_b = \frac{17 + 3\sqrt{3}}{24} d^2$

$\Rightarrow \eta = \frac{24}{17 + 3\sqrt{3}} \approx 1.08$

This constellation is <u>less</u> efficient than regular 16-QAM

4) P. 3.17

a)



b) There are 4 typical signals, each occurs 4 times.

$$\Rightarrow P_e = \frac{1}{4}\left(P_{e|s_1} + P_{e|s_2} + P_{e|s_3} + P_{e|s_4}\right)$$

$$E_s = \frac{1}{4}\cdot\left(2 + 9 + 18 + 25\right) = \frac{27}{2} \Rightarrow E_b = \frac{27}{8}$$

$$d_{min} = 2 \qquad \Rightarrow \eta = \frac{d_{min}^2}{E_b} = \frac{4}{27/8} = \frac{32}{27}$$

For each of the typical signals, we enumerate the distance to neighbors that share a decision boundary:

S_1 has neighbors at distances: $2, \sqrt{13}, \sqrt{13}$
S_2 has neighbors at: $2, 3, 3, \sqrt{5}, \sqrt{5}$
S_3 has neighbors at: $\sqrt{8}, 3, 3, \sqrt{13}, \sqrt{13}$
S_4 has neighbors at: $2, 2, \sqrt{5}, \sqrt{5}, \sqrt{8}$

Each neighbor at distance d contributes a term of the form:

$$Q\left(\frac{d}{\sqrt{2N_0}}\right) = Q\left(\frac{d}{d_{min}} \cdot \sqrt{\frac{d_{min}^2}{E_b}} \cdot \sqrt{\frac{E_b}{2N_0}}\right)$$
$$= Q\left(\frac{d}{d_{min}} \cdot \sqrt{\eta \cdot \frac{E_b}{2N_0}}\right)$$

Thus:

$$P_{e|s_1} \le Q\left(\sqrt{\frac{16E_b}{27N_0}}\right) + 2Q\left(\sqrt{\frac{13}{4}}\sqrt{\frac{16E_b}{27N_0}}\right)$$

$$P_{e|s_2} \le Q\left(\sqrt{\frac{16E_b}{27N_0}}\right) + 2Q\left(\frac{3}{2}\sqrt{\frac{16E_b}{27N_0}}\right) + 2Q\left(\sqrt{\frac{5}{4}}\sqrt{\frac{16E_b}{27N_0}}\right)$$

$$P_{e|s_3} \le Q\left(\sqrt{2}\sqrt{\frac{16E_b}{27N_0}}\right) + 2Q\left(\frac{3}{2}\sqrt{\frac{16E_b}{27N_0}}\right) + 2Q\left(\sqrt{\frac{13}{4}}\sqrt{\frac{16E_b}{27N_0}}\right)$$

$$P_{e|s_4} \le 2Q\left(\sqrt{\frac{16E_b}{27N_0}}\right) + 2Q\left(\sqrt{\frac{5}{4}}\sqrt{\frac{16E_b}{27N_0}}\right) + Q\left(\sqrt{2}\sqrt{\frac{16E_b}{27N_0}}\right)$$

After averaging and "cleaning up":

$$P_e \le Q\left(\sqrt{\frac{16E_b}{27N_0}}\right) + \frac{1}{2}Q\left(\sqrt{\frac{32E_b}{27N_0}}\right) + Q\left(\sqrt{\frac{20E_b}{27N_0}}\right) + Q\left(\sqrt{\frac{4E_b}{3N_0}}\right) + Q\left(\sqrt{\frac{52E_b}{27N_0}}\right)$$

c) Nearest neighbor approximation:

$$P_e \approx Q\left(\sqrt{\frac{16E_b}{27N_0}}\right)$$