ECE 630: Statistical Communication Theory
Prof. B.-P. Paris
Homework 4
Due: February 26, 2019

Reading Madhow: Section 3.3.

Problems

1. The stationary random process $X_t$ is passed through a linear filter with transfer function $H(f)$,

$$H(f) = \frac{j2\pi f + a}{j2\pi f + 2a}.$$

The output process is labeled $Y_t$. The mean of $Y_t$ is measured to be $\frac{1}{2}$ and the covariance function of $Y_t$ is found to be

$$K_Y(\tau) = a^2 e^{-2a|\tau|}.$$

(a) Compute the power spectral density of $Y_t$.
(b) Find the second order description of $X_t$.

2. In practice one often wants to measure the power spectral density of a stochastic process. For the purposes of this problem, assume the process $X_t$ is wide-sense stationary, zero mean, and Gaussian. The following measurement system is proposed.

Here $H_1(f)$ is the transfer function of an ideal bandpass filter and $H_2(f)$ is an ideal lowpass,

$$H_1(f) = \begin{cases} 
1 & \text{for } f_0 - \frac{\Delta f}{2} \leq |f| \leq f_0 + \frac{\Delta f}{2} \\
0 & \text{else}
\end{cases}$$

$$H_2(f) = \begin{cases} 
\frac{1}{2\pi f} & \text{for } |f| \leq \Delta f \\
0 & \text{else}
\end{cases}$$

Assume that $\Delta f$ is small compared to the range of frequencies over which $S_X(f)$ varies, i.e., you may assume that $S_X(f)$ is constant over intervals of width $\Delta f$.

(a) Find the mean and correlation function of $Y_t^2$ in terms of the second order description of $X_t$. The following may be helpful — this is known as Isserlin’s Theorem: If $X$ and $Y$ are jointly Gaussian, then $E[X^2Y^2] = E[X^2]E[Y^2] + 2E^2[XY]$
(b) Compute the power spectral density of the process \( Z_t \).
(c) Compute the expected value of \( Z_t \).
(d) By considering the variance of \( Z_t \), comment on the accuracy of this measurement of the power density of the process \( X_t \).

3. Let \( W_t \) (for \( t \geq 0 \)) be a Wiener process (Brownian motion) with variance \( \sigma^2 \). Define the random process \( X_t \) as the (running) integral over \( W_t \), i.e., for \( t \geq 0 \)

\[
X_t = \int_0^t W_s \, ds.
\]

(a) Find the mean of \( X_t \).
(b) Compute the autocorrelation function of \( X_t \).
(c) Is \( W_t \) wide-sense stationary?
(d) Compute the following probability for \( t \geq 0 \)

\[
Pr\{|X_t| > \sigma \cdot t\}.
\]