1. A stochastic process is defined by

\[ X_t = \cos(2\pi Ft) \]

where the frequency \( F \) is uniformly distributed over the interval \([0, f_0]\).

(a) Find the mean and correlation function of \( X_t \).
(b) Show that this process is non-stationary.

Now suppose we redefine the process \( X_t \) to be

\[ X_t = \cos(2\pi Ft + \Theta) \]

where \( F \) and \( \Theta \) are statistically independent random variables. \( \Theta \) is uniformly distributed over \([-\pi, \pi]\) and \( F \) is distributed as before.

(c) Compute the mean and correlation function of \( X_t \).
(d) Is \( X_t \) wide-sense stationary? Show your reasoning.
(e) Find the first order density \( p_{X_t}(x) \).

2. The stationary random process \( X_t \) is passed through a linear filter with transfer function \( H(f) \),

\[ H(f) = \frac{j2\pi f + a}{j2\pi f + 2a}. \]

The output process is labeled \( Y_t \). The mean of \( Y_t \) is measured to be \( \frac{1}{2} \) and the covariance function of \( Y_t \) is found to be

\[ K_Y(\tau) = a^2 e^{-2\alpha|\tau|}. \]

(a) Compute the power spectral density of \( Y_t \).
(b) Find the second order description of \( X_t \).

3. In practice one often wants to measure the power spectral density of a stochastic process. For the purposes of this problem, assume the process \( X_t \) is wide-sense stationary, zero mean, and Gaussian. The following measurement system is proposed.
Here $H_1(f)$ is the transfer function of an ideal bandpass filter and $H_2(f)$ is an ideal lowpass,

$$H_1(f) = \begin{cases} 
1 & \text{for } f_0 - \frac{\Delta f}{2} \leq |f| \leq f_0 + \frac{\Delta f}{2} \\
0 & \text{else}
\end{cases}$$

$$H_2(f) = \begin{cases} 
\frac{1}{\pi \Delta f} & \text{for } |f| \leq \Delta f \\
0 & \text{else}
\end{cases}$$

Assume that $\Delta f$ is small compared to the range of frequencies over which $S_X(f)$ varies, i.e., you may assume that $S_X(f)$ is constant over intervals of width $\Delta f$.

(a) Find the mean and correlation function of $Y_t^2$ in terms of the second order description of $X_t$.

(b) Compute the the power spectral density of the process $Z_t$.

(c) Compute the expected value of $Z_t$.

(d) By considering the variance of $Z_t$, comment on the accuracy of this measurement of the power density of the process $X_t$.

4. Let $W_t$ (for $t \geq 0$) be a Wiener process (Brownian motion) with variance $\sigma^2$. Define the random process $X_t$ as the (running) integral over $W_t$, i.e., for $t \geq 0$

$$X_t = \int_0^t W_s \, ds.$$ 

(a) Find the mean of $X(t)$.

(b) Compute the autocorrelation function of $X_t$.

(c) Is $W_t$ wide-sense stationary?

(d) Compute the following probability for $t \geq 0$

$$\Pr\{|X_t| > \sigma \cdot t\}.$$